國 立 政 治 大 學 應 用 數 學 系

碩 士 學 位 論 文

向右之具長域 Domany-Kinzel 模型的漸進行為

Asymptotic behavior for a long-range Domany-Kinzel model with right direction

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中文摘要

在本篇文章中，我們介紹一種向右之具長域的 Domany-Kinzel 模型，其模型定義在二維方格座標上，假設 n 為一個非負整數，每個座標點 (a,b) 都擁有具機率一的向右有向鏈結，並擁有 $n+1$ 個分別具有 $p_k \in (0,1)$ 機率的從 (a,b) 到 $(a+k,b+1)$ 之有向鏈結，其中 $a,b \in \mathbb{Z}_+$ 且 $k=0,1,\cdots,n$。假設 $\tau_n(N,M)$ 為從 $(0,0)$ 到 (N,M) 至少有一個由被滲透的邊組成之連通的有向路徑之機率，定義長寬比以 $\alpha = N/M$ 表示，我們求得臨界值 $\alpha_{n,c} \in \mathbb{R}_+$ 使得當 $\alpha = \alpha_{n,c}$ 時在 M 趨近於無限下 $\tau_n(N,M)$ 趨近於 1/2，並對其收斂速率進行研討。進而我們研究對 n 趨近於無限時模型的表現，在 m 為非負整數且 $p_m \in [0,1)$ 的前提下，特別聚焦於 $p_m \approx_{m\to\infty} \frac{p}{m^s}$ 其中 $p \in (0,1)$、$s > 1$，以及 $p_m = \frac{e^{-\lambda}\lambda^m}{m!}$ 其中 $\lambda > 0$ 這兩種假設情況進行討論，我們發現當 s 和 λ 的值符合前述情境時，$\lim_{n\to\infty} \tau_n(N,M)$ 的極值表現與先前 n 為非負整數時的結果相似，並且在 n 趨近於無限的模型中，$\lim_{n\to\infty} \tau_n(N,M)$ 的極值表現受 α 逼近 $\alpha_{n,c}$ 的速度影響甚劇。

關鍵字：Domany-Kinzel 模型、定向滲流、隨機漫步、漸進行為、臨界值行為、Berry-Esseen 定理、大離差定理

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Abstract

In this thesis, we introduce a certain type of Domany-Kinzel model which may be regarded as a long-range model with right direction in two-dimension rectangular lattices. For a fixed non-negative integer n, every site (a, b) possesses not only a directed bond from site (a, b) to $(a + 1, b)$ with probability one but also $n + 1$ directed bonds from (a, b) to $(a + k, b + 1)$ with respectively probabilities $p_k \in (0, 1), \forall a, b \in \mathbb{Z}_+, k = 0, 1 \cdots n$. Let $\tau_n(N, M)$ be the probability that there is at least one connected-directed path of occupied edges from $(0, 0)$ to (N, M) and let α be the aspect ratio which means $\alpha = N/M$. We conclude that $\tau_n(N, M)$ converges to 1, 0, and 1/2 as $M \to \infty$ for $\alpha > \alpha_{n,c}$, $\alpha < \alpha_{n,c}$, and $\alpha = \alpha_{n,c}$, respectively, where $\alpha_{n,c} \in \mathbb{R}_+$ is the critical value. The rate of convergence is discussed, too. Moreover, we study the cases that n tends to infinity. Specifically, for $p_m \in [0, 1)$ with $m \in \mathbb{Z}_+$, we discuss the two cases in detail which are $p_m \approx_{m\to\infty} \frac{p}{m^s}$ with $p \in (0, 1)$, $s > 1$ and $p_m = \frac{e^{-\lambda}\lambda^m}{m!}$ with $\lambda > 0$. We discover that the behavior of $\lim_{n\to\infty} \tau_n(N, M)$ is similar to the case that n is a non-negative integer when s and λ fit the definition. Moreover, the speed of α approaching to the critical apect ratio highly influences the behavior of $\lim_{n\to\infty} \tau_n(N, M)$.

Keywords: Domany-Kinzel model, directed percolation, random walk, asymptotic behavior, critical behavior, Berry-Esseen theorem, large deviation.

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Chapter 1

Introduction

In 1957, percolation and directed percolation were first studied by Broadbent and Hammersley [2]. For the reseachers in probability and statistical mechanics, these topics are still one of the most extraordinary fascinating problems in this day. With referring to $[1, 11]$ and their references, we can find numerous properties, conjectures and results of percolation and directed percolation. But not much is known about the exact solutions to the percolation problem, especially to the directed percolation problems.

In 1981, a doable version of directed percolation on the square lattice was defined by Domany and Kinzel [8] as follows. Each vertical bond is directed upward with occupation probability p, where p is a fixed real number in $(0, 1)$, and each horizontal bond is directed rightward with occupation probability 1. An essential point of this solvable version is that the occupation probability of every single bond is independent with each other. Moreover, the boundary of the Domany-Kinzel model is known to have the same distribution as the one-dimensional last passage percolation model (see [10]).

Since the model mentioned above is highly specialized, it has been considered in more general cases recently. There are several directed percolation models that are considered on the square lattice. For instance, the model's horizontal edges are occupied with probability 1 in the even rows and $p_h \in [0, 1]$ in the odd rows while the vertical edges are defined with occupation probability $p_v \in (0, 1)$ [7]; the model's horizontal edges are defined with occupation probability 1 while the vertical edges in the n-th column are occupied with probabilities $p_1 \in [0, 1),\ p_2 \in [0, 1)$ alternatively if n is even and probabilities $p_2,\ p_1$ alternatively if n is odd [6]. Furthermore, there are some directed percolation models that are considered on more complicated lattices such

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as triangular lattice and honeycomb lattice. For example, the model on the triangular lattice whose horizontal edges are directed rightward with occupation probabilities 1 and $x \in [0,1]$ alternatively, vertical edges are directed upward with occupation probability $y \in (0,1)$, and diagonal edges from lower-left to upper-right or from lower-right to upper-left with occupation probability $d \in [0,1)$ [3]; the model on the honeycomb lattice as bricks whose horizontal edges are directed rightward with occupation probabilities 1 and $x \in [0,1]$ in alternate rows while vertical edges are directed upward with occupation probability $y \in (0,1)$ [4].

In 1983 [13], Li and Zhang introduce the long-range Domany-Kinzel model with left direction on the two-dimensional lattice as follows. For every site (a,b), where $a,b \in \mathbb{Z}_+$, there is a directed bond present from site (a,b) to $(a+1,b)$ with probability 1. There are also $n+1$ directed bonds present from (a,b) to $(a-k+1,b+1)$, $k=0,1,2,\cdots,n$ with respective probabilities $p_k \in (0,1)$ where $n \in \mathbb{Z}_+$. They also obtained the limiting behavior of this model. The model had been extended to a more general case by Chang and Chen in 2018 [5]. Moreover, they obtained the asymptotic behavior of it.

In this thesis, we introduce a certain type of Domany-Kinzel model on the two-dimensional lattice which may be regarded as a long-range model with right direction, instead of left direction, and we define the model as follows. For every site (a,b), where $a,b \in \mathbb{Z}_+$, there is a directed bond present from site (a,b) to $(a+1,b)$ with probability 1 and there are $n+1$ directed bonds present from (a,b) to $(a+k,b+1)$, with respectively probabilities $p_k \in (0,1)$, $k=0,1,\cdots,n$, where n is any non-negative integer (see Fig 1.1).

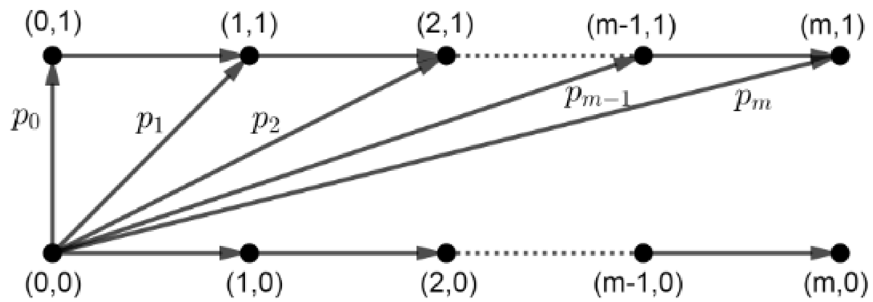

Figure 1.1: The long-range right directed bond Domany-Kinzel model.

Throughout this thesis, we define some notations as follows. We denote ${p'}_k = 1 - p_k$ for $k=0,1,\cdots,n$ and $\bar{p'}_n = \prod_{j=0}^{n} {p'}_j$. $f(M) \approx_{M\to\infty} g(M)$ means that $\lim_{M\to\infty} f(M)/g(M) \in (0,\infty)$. Similarly, for a fixed α_0, $f(\alpha) \approx_{\alpha\to\alpha_0} g(\alpha)$ means that $\lim_{\alpha\to\alpha_0} f(\alpha)/g(\alpha) \in (0,\infty)$.

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The vertex (a, b) is said to be percolating if there is at least one connected-directed path of occupied edges from $(0,0)$ to (a, b). And the notation $(0,0) \rightsquigarrow (a, b)$ is used to denote that (a, b) is percolating in this thesis. For any $\alpha \in \mathbb{R}$, denote $M_\alpha = \lfloor \alpha M \rfloor = \sup\{i \in \mathbb{Z} : i \leq \alpha M\}$ with non-negative integer M. Let $\mathbb{P}$ be the probability distribution of the bond variables, and define the two point correlation function, with respect to n,

$$\tau_n(M_\alpha, M) = \mathbb{P}\big((0,0) \rightsquigarrow (M_\alpha, M)\big) . \tag{1.1}$$

For triangle lattices (in our model is n=1), it was shown in [15] by the method of steepest descent that there is

$$\alpha_{1,c} = p'_0 + \frac{p'^2_0 p'_1}{1 - p'_0 p'_1} ,$$

such that

$$\lim_{M\to\infty} \tau_1(M_\alpha, M) = \begin{cases} 1, \text{ if } \alpha > \alpha_{1,c} , \\ 0, \text{ if } \alpha < \alpha_{1,c} , \\ \frac{1}{2}, \text{ if } \alpha = \alpha_{1,c} . \end{cases} \tag{1.2}$$

Remark 1.1. *For $n = 0$, it was shown that $\alpha_{0,c} = \frac{p'_0}{1-p'_0}$.*

In fact, probability theory is extremely effective to deal with this model. More specifically, we can get $\alpha_{1,c}$ and the result of (1.2) easily by the law of large number rather than the method of steepest descent. Moreover, we can extend the result in (1.2) to all $n \in \mathbb{Z}_+$ in this thesis (see Theorem 2.1 in the next chapter). Note that it is square lattices for $n = 0$ and triangle lattices for $n = 1$. The limiting behavior of two-point function in (1.2) is really interesting as the critical point is discontinuous.

It is appropriate to define some of the standard critical exponents and to sketch the phenomenological scaling theory of $\tau_n(M_\alpha, M)$. For α near $\alpha_{n,c}$, the scaling theory of critical behavior now asserts that the singular part of $\tau_n(M_\alpha, M)$ varies asymptotically as (see [12])

$$\tau_n(M_\alpha, M) \sim \frac{A_\alpha}{M^\mu} \exp\left(\frac{-B_\alpha M}{(\alpha_{n,c} - \alpha)^{-\nu}}\right) , \tag{1.3}$$

where $f(M) \sim g(M)$ means that $\lim_{M\to\infty} f(M)/g(M) = 1$, the constants A_α and B_α depend on α, and $\mu, \nu \in (0, \infty)$ are universal constants. Furthermore, μ is called the critical exponent

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and ν is called the critical exponent of the correlation length [13]. Note that there has been no general proof of the existence of critical exponents. This allows us analyze (1.2) in detail. In this thesis, we use the Berry-Esseen theorem and large deviation argument to investigate the asymptotic behavior of $\tau_n(M_\alpha, M)$ for any $n \in \mathbb{Z}_+$ and as $n \to \infty$.

The rest of this thesis is organized as follows. The main results are presented in chapter 2. In chapter 3, we describe how we derive $\alpha_{n,c}$ and σ_n^2 in section 3.2, and 3.3, respectively. Moreover, we describe the behavior of $\alpha_{n,c}$ and σ_n^2 as $n \to \infty$ in 3.4. The proofs of the main results are presented in chapter 4.

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Chapter 2

Main results

In this chapter, we present the main theorems of this thesis. As beginning, we study the model in the case that n is a finite non-negative integer. In this case, the critical aspect ratio, $\alpha_{n,c}$, and the rate of convergence are derived. Then we study the case $n \to \infty$. With different assumptions of p_k, the limit behavior of $\tau_n(M_\alpha, M)$ are learned. Finally, we investigate the asymptotic behavior of $\tau_n(M_\alpha, M)$ in the large M limit and $n \to \infty$. Notice that the differences between [5] and this thesis are mentioned in the remarks after the theorems.

Theorem 2.1. *Given a finite $n \in \mathbb{Z}_+$ and $p_k \in (0,1)$, $k = 0, 1, 2, \cdots, n$, there is a critical aspect ratio*

$$\alpha_{n,c} = \sum_{j=1}^{n} {p'}_0^j {p'}_1^{j-1} \cdots {p'}_{j-1} + \frac{1}{1 - \bar{p}'_n} \prod_{l=0}^{n} {p'}_l^{n+1-l}, \tag{2.1}$$

such that, in the large M limit,

$$\tau_n(M_{\alpha_{n,c}}, M) = \frac{1}{2} + O(\frac{1}{\sqrt{M}}) \tag{2.2}$$

and when α is close to $\alpha_{n,c}$ but not equal to $\alpha_{n,c}$

$$\tau_n(M_\alpha, M) \le e^{-MI(\alpha)} \textit{ for } \alpha < \alpha_{n,c} \,,$$
$$1 - \tau_n(M_\alpha, M) \le e^{-MI(\alpha)} \textit{ for } \alpha > \alpha_{n,c} \,,$$

where

$$I(\alpha) \approx_{\alpha \to \alpha_{n,c}} (\alpha - \alpha_{n,c})^2 \,. \tag{2.3}$$

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Notice that the rate function $I(\alpha)$ in (2.3) is optimal [14], which gives the upper bound of $\tau_n(M_\alpha, M)$ or $1-\tau_n(M_\alpha, M)$ as $e^{-MI(\alpha)}$ when α is smaller or larger than $\alpha_{n,c}$, respectively.

Remark 2.1. *For the long-range Domany-Kinzel model with left direction in [5], we have the critical aspect ratio*

$$\alpha_{n,c} = -\sum_{j=2}^{n}(1 - p'_j p'^2_{j+1} \cdots p'^{n-j+1}_n) + \frac{1}{1-\bar{p'}_n}\prod_{l=1}^{n} p'^l_l$$

which has several differences to (2.1) and this leads to inconsistent results in the following theorems with those of [5].

Since researching into the limit behavior as $n \to \infty$ brings quite a sense of interest, the following passages of this chapter is dedicated to it. For convenience, we define

$$\tau(M_\alpha, M) = \mathbb{P}\big((0,0) \rightsquigarrow (M_\alpha, M)\big) \tag{2.4}$$

to represent $\tau_n(M_\alpha, M)$ in the case $n \to \infty$.
It's essential that $p_k \to 0$ as $k \to \infty$, we rewrite

$$\bar{p'}_n = e^{\sum_{k=0}^{n} \log(1-p_k)} \text{ and } p'^j_0 p'^{j-1}_1 \cdots p'_{j-1} = e^{\sum_{k=0}^{j-1}(j-k)\log(1-p_k)}. \tag{2.5}$$

By Maclaurin series of $log(1-x)$ where $x < 1$,

$$-p_k(1+p_k) \leq \log(1-p_k) \leq -p_k \text{ for } p_k \in (0, 0.6838026238)\ ,$$

so we can get the approximations

$$\bar{p'}_n \approx_{n\to\infty} e^{-\sum_{k=0}^{n} p_k} \tag{2.6}$$

and for $j \geq 1$

$$p'^j_0 p'^{j-1}_1 \cdots p'_{j-1} \approx_{n\to\infty} e^{-\sum_{k=0}^{j-1}(j-k)p_k} \tag{2.7}$$

Hence, under the condition $p_k \to 0$ as $k \to \infty$, we obtain from (2.1) that

$$\alpha_c := \lim_{n\to\infty} \alpha_{n,c} \approx \lim_{n\to\infty} \big(e^{-\sum_{k=0}^{j-1}(j-k)p_k} + \frac{e^{-\sum_{k=0}^{n}(n+1-k)p_k}}{1-e^{-\sum_{k=0}^{n} p_k}}\big)\ . \tag{2.8}$$

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As a result of convenience, the subscript n is omitted in the case $n \to \infty$ henceforth.

The following two cases for $n \to \infty$ deserve further consideration.

(i) $p_m \approx_{m\to\infty} \frac{p}{m^s}$ with $p \in (0,1)$, $s > 0$;

(ii) $p_m = \frac{e^{-\lambda}\lambda^m}{m!}$ with $\lambda > 0$ (note that $p_m \in [0,1)$ with $m \in \mathbb{Z}_+$).

The behaviors of these two cases are illustrate in Theorem 2.2 and Theorem 2.3, respectively.

Theorem 2.2. *Let $p_m \in [0,1)$ with $m \in \mathbb{Z}_+$ and $p_m \approx_{m\to\infty} \frac{p}{m^s}$ with $p \in (0,1)$, $s > 0$. For $s > 1$, we have*

$$\alpha_c \in (0,\infty), \quad \sigma \in (0,\infty), \tag{2.9}$$

and, in the large M limit,

$$\tau(M_{\alpha_c}, M) = \frac{1}{2} + O(\frac{1}{\sqrt{M}}), \quad s > 1. \tag{2.10}$$

Furthermore, for s>1, when α is close to α_c but not equal to α_c, we have

$$\begin{aligned} \tau(M_\alpha, M) &\le e^{-MI(\alpha)} \quad \textit{for } \alpha < \alpha_c, \\ 1 - \tau(M_\alpha, M) &\le e^{-MI(\alpha)} \quad \textit{for } \alpha > \alpha_c, \end{aligned} \tag{2.11}$$

where

$$I(\alpha) \approx (\alpha - \alpha_c)^2. \tag{2.12}$$

Remark 2.2. *For the long-range Domany-Kinzel model with left direction in [5] with the same assumption in Theorem 2.2, $\alpha_c \in (-\infty, 1)$ for $s > 3$ and $\sigma^2 \in (0,\infty)$ for $s > 4$ and these lead to different limit behavior of $\tau(M_{\alpha_c}, M)$ (see [5] Theorem 2.4).*

Theorem 2.3. *Let $p_m \in [0,1)$ with $m \in \mathbb{Z}_+$ and $p_m = \frac{e^{-\lambda}\lambda^m}{m!}$ with $\lambda > 0$. For $\lambda > 0$, we have*

$$\alpha_c \in (0,\infty), \quad \sigma \in (0,\infty), \tag{2.13}$$

and, in the large M limit,

$$\tau(M_{\alpha_c}, M) = \frac{1}{2} + O(\frac{1}{\sqrt{M}}). \tag{2.14}$$

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Furthermore, when α is close to α_c but not equal to α_c, we have

$$\begin{aligned} \tau(M_\alpha, M) &\leq e^{-MI(\alpha)} \quad \textit{for } \alpha < \alpha_c, \\ 1 - \tau(M_\alpha, M) &\leq e^{-MI(\alpha)} \quad \textit{for } \alpha > \alpha_c, \end{aligned} \tag{2.15}$$

where

$$I(\alpha) \approx (\alpha - \alpha_c)^2. \tag{2.16}$$

The following theorem is dealing with the asymptotic behavior of $\tau(M_\alpha, M)$ in the large M limit.

Theorem 2.4. *Given $p_m \in [0,1)$ with $m \in \mathbb{Z}_+$ and $p_m \approx_{m\to\infty} \frac{p}{m^s}$ with $p \in (0,1)$ and $s > 1$, let $\alpha_M^- = \alpha_c - M^{\frac{\rho}{2}\ell_M}$ and $\alpha_M^+ = \alpha_c + M^{\frac{\rho}{2}\ell_M}$, where $\rho \in (0,\infty)$ and $\{\ell_M\}_{M=1}^\infty$ is a positive slowly varying sequence. Note that the positive slowly varying sequence is defined as $\forall M \in \mathbb{N}$ $\ell_M > 0$ and $\lim_{M\to\infty} \frac{\ell_{aN}}{\ell_M} = 1 \; \forall a > 0$. We obtain the following results in the large M limit.*

$$\tau(M_{\alpha_M^-}, M), 1 - \tau(M_{\alpha_M^+}, M) =$$

$$\begin{cases} O(1) \max\left\{\frac{\sigma}{M^{\frac{1-\rho}{2}}\ell_M} e^{-\frac{M^{1-\rho}\ell_M^2}{2\sigma^2}}, \frac{1}{\sqrt{M}}\right\} & \textit{, if } \rho \in (0,1), \\ \Psi(\frac{L}{\sigma}) + O(1) \max\left\{|\ell_M - L|, \frac{1}{\sqrt{M}}\right\}, & \textit{, if } \rho = 1, \textit{ and } \lim_{M\to\infty} \ell_M = L \in (0,\infty), \\ O(1) \max\left\{\frac{\sigma}{\ell_M} e^{-\frac{\ell_M^2}{2\sigma^2}}, \frac{1}{\sqrt{M}}\right\} & \textit{, if } \rho = 1, \textit{ and } \lim_{M\to\infty} \ell_M = \infty, \\ \frac{1}{2} + O\left(\frac{\ell_M}{M^{\frac{\rho-1}{2}}}\right), & \textit{, if } \rho \in (1,2], \\ \frac{1}{2} + O\left(\frac{1}{\sqrt{M}}\right), & \textit{, if } \rho > 2. \end{cases} \tag{2.17}$$

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Chapter 3

Random walk

3.1 Derivation of D_n

For any $M \in \mathbb{N}$, an occupied vertical edge in a bond configuration is said to be wet if it lies on a percolating path where (M_α, M) is percolating. A certain occupied vertical edge ending at (k, m) is said to be *primary* wet if it is the wet edge with smallest k value for that m. In a percolating configuration where (M_α, M) is percolating, there is only one *primary wet* edge for each $m \in \{1, 2, \cdots, M\}$. Define $C_{n,M}(k)$ as the probability that the *primary wet* edge for $m = M$ ending at (k, M), and let us formally define $C_{n,0}(k) = \delta_{0,k}$ where δ is the Kronecker delta, i.e.,

$$\delta_{0,k} = \begin{cases} 1, \text{ if } k = 0\ , \\ 0, \text{ if } k \neq 0\ . \end{cases}$$

Since the primary wet edge can occur at any value of $k \leq M_\alpha$, we obtain

$$\tau_n(M_\alpha, M) = \sum_{k \leq M_\alpha} C_{n,M}(k) \tag{3.1.1}$$

for $M \in \mathbb{N}$.

By the definition of the model in this thesis and the properties of random walk, for any integer k and non-negative integer m we have

$$C_{n,m+1}(k) = \sum_{j=0}^{k} C_{n,m}(k-j) D_n(j), \tag{3.1.2}$$

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where

$$D_n(j) = \begin{cases} 0 & \text{, if } j < 0\,, \\ 1 - p'_0 & \text{, if } j = 0\,, \\ (p'^j_0 p'^{j-1}_1 \cdots p'_{j-1})(1 - p'_0 p'_1 \cdots p'_{j-1} p'_j) & \text{, if } 1 \le j \le n\,, \\ (\bar{p'}_n)^{j-n} (p'^n_0 p'^{n-1}_1 \cdots p'_{n-1})(1 - \bar{p'}_n) & \text{, if } j > n\,. \end{cases} \tag{3.1.3}$$

For convenience, define $U_n(j) = (p'^j_0 p'^{j-1}_1 \cdots p'_{j-1})(1 - p'_0 p'_1 \cdots p'_{j-1} p'_j)$, if $0 \le j \le n$. As an example, we illustrate $D_3(1)$ in Figure 3.1 and $D_3(2)$ in Figure 3.2 for $n = 3$.

In the case of $D_3(1)$, we mention two paths in Figure 3.1, that is (a) and (b). The connection with an arrow is occupied, while the others with double bar is unoccupied. The dotted lines can be either occupied or unoccupied. The probabilities of the two paths, (a) $P(\{(0,0) \to (1,1)\}) = p'_0 p_1$ and (b) $P(\{(1,0) \to (1,1)\}) = p'_0 p'_1 p_0$, lead us to the operation of $D_3(1)$ in the following.

$$\begin{aligned} D_3(1) &= P(\{(0,0) \to (1,1)\}) + P(\{(1,0) \to (1,1)\}) \\ &= p'_0 p_1 + p'_0 p'_1 p_0 \\ &= p'_0 (1 - p'_1 + p'_1 (1 - p'_0)) \\ &= p'_0 (1 - p'_0 p'_1)\,. \end{aligned}$$

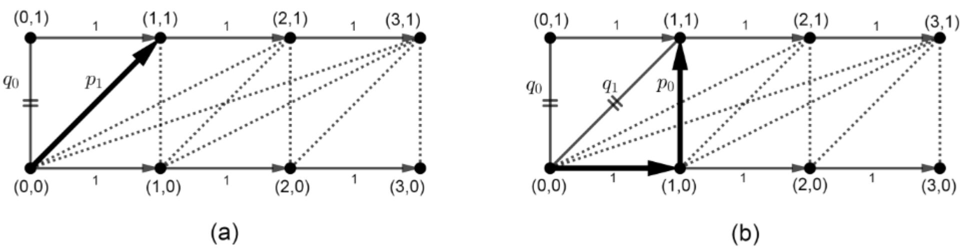

Figure 3.1: The illustration for $D_3(1)$.

In the case of $D_3(2)$, we mention three paths in Figure 3.2, that is (a), (b), and (c). The connection with an arrow is occupied, while the others with double bar is unoccupied. The dotted lines can be either occupied or unoccupied. The probabilities of the three paths, (a)

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$P(\{(0,0) \to (2,1)\}) = p'^2_0 p'_1 p_2$, (b) $P(\{(1,0) \to (2,1)\}) = p'^2_0 p'_1 p'_2 p_1$, and (c) $P(\{(2,0) \to (2,1)\}) = p'^2_0 p'^2_1 p'_2 p_0$, lead us to the operation of $D_3(1)$ in the following.

$$\begin{aligned} D_3(2) &= P(\{(0,0) \to (2,1)\}) + P(\{(1,0) \to (2,1)\}) + P(\{(2,0) \to (2,1)\}) \\ &= p'^2_0 p'_1 p_2 + p'^2_0 p'_1 p'_2 p_1 + p'^2_0 p'^2_1 p'_2 p_0 \\ &= p'^2_0 p'_1 [1 - p'_2 + p'_2(1 - p'_1) + p'_1 p'_2 (1 - p'_0)] \\ &= p'^2_0 p'_1 (1 - p'_0 p'_1 p'_2) \end{aligned}$$

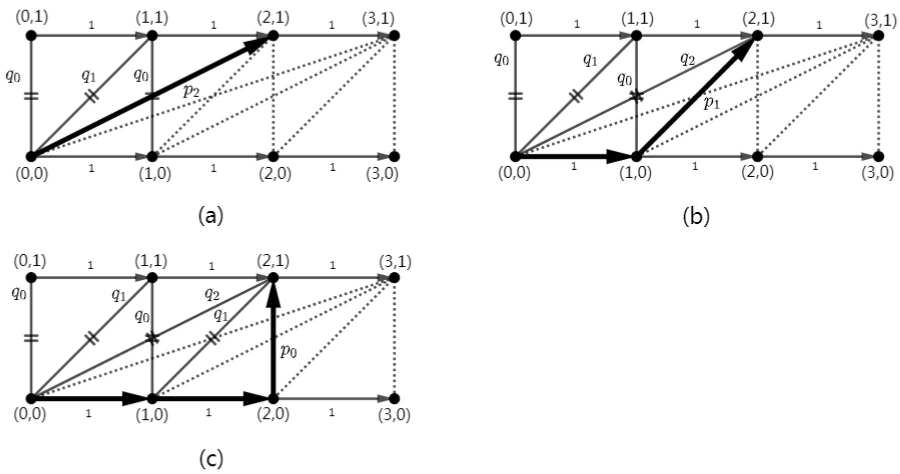

Figure 3.2: The illustration for $D_3(2)$.

3.2 Derivation of $\alpha_{n,c}$

The generating function of a probability distribution $f : \mathbb{Z}_+ \to \mathbb{R}_+$ can be defined as

$$\hat{f}(t) = \sum_{j \in \mathbb{Z}} f(j) t^j \text{ , where } |t| \text{ is less than the radius of convergence.} \tag{3.2.1}$$

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Note that

$$\begin{aligned}\hat{C}_{n,m}(t) &= \sum_{k=0}^{\infty} C_{n,m}(k)t^k \\ &= \sum_{k=0}^{\infty} \Big[\sum_{j=0}^{k} C_{n,m-1}(k-j)D_n(j)\Big] t^{k-j}t^j \\ &= \sum_{j\in\mathbb{Z}} D_n(j)t^j \sum_{k=j}^{\infty} C_{n,m-1}(k-j)t^{k-j} \quad \text{by Fubini's thm} \\ &= \sum_{j\in\mathbb{Z}} D_n(j)t^j \sum_{k=0}^{\infty} C_{n,m-1}(k)t^k \\ &= \hat{D}_n(t)\hat{C}_{n,m-1}(t)\,.\end{aligned} \tag{3.2.2}$$

By repeating (3.2.2) m times, it follows that

$$\hat{C}_{n,m}(t) = \hat{D}_n(t)^m\,. \tag{3.2.3}$$

By (3.1.3), we have

$$\begin{aligned}\hat{D}_n(t) &= \sum_{j\in\mathbb{Z}} D_n(j)t^j \\ &= \sum_{j\in\mathbb{Z},j<0} D_n(j)t^j + (1-p'_0)t^0 + \sum_{j=1}^{n} U_n(j)t^j + \sum_{j\in\mathbb{Z},j>n} (\bar{p'}_n)^{j-n}(\prod_{l=0}^{n-1} p'^{n-l}_l)(1-\bar{p'}_n)t^j \\ &= 0 + (1-p'_0) + \hat{U}_n(t) + (\prod_{l=0}^{n-1} p'^{n-l}_l)(1-\bar{p'}_n)(\sum_{j=1}^{\infty}(\bar{p'}_n)^{j-n}t^j - \sum_{j=1}^{n}(\bar{p'}_n)^{j-n}t^j) \\ &= (1-p'_0) + \hat{U}_n(t) + (\prod_{l=0}^{n-1} p'^{n-l}_l)(1-\bar{p'}_n)(\frac{(\bar{p'}_n)^{1-n}t}{1-\bar{p'}_n t} - \frac{(\bar{p'}_n)^{1-n}t[1-(\bar{p'}_n t)^n]}{1-\bar{p'}_n t}) \\ &= (1-p'_0) + \hat{U}_n(t) + (\prod_{l=0}^{n-1} p'^{n-l}_l)(1-\bar{p'}_n)(\frac{\bar{p'}_n t^{n+1}}{1-\bar{p'}_n t})\,,\end{aligned} \tag{3.2.4}$$

where

$$\hat{U}_n(t) = \sum_{j=1}^{n} (p'^j_0 p'^{j-1}_1 \cdots p'_{j-1})(1 - p'_0 p'_1 \cdots p'_{j-1} p'_j)t^j\,.$$

Note that

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$$
\begin{aligned}
1-p'_0+\hat{U}_n(1) &= 1-p'_0+\sum_{j=1}^{n}(p'^{j}_0 p'^{j-1}_1\cdots p'_{j-1})(1-p'_0p'_1\cdots p'_{j-1}p'_j) \\
&= 1-p'_0+\sum_{j=1}^{n}p'^{j}_0 p'^{j-1}_1\cdots p'_{j-1}-\sum_{j=1}^{n}p'^{j+1}_0 p'^{j}_1\cdots p'^{2}_{j-1}p'_j \\
&= 1-p'_0+\sum_{j=1}^{n}p'^{j}_0 p'^{j-1}_1\cdots p'_{j-1}-\sum_{j=2}^{n+1}p'^{j}_0 p'^{j-1}_1\cdots p'_{j-1} \\
&= 1-p'_0+p'_0+\sum_{j=2}^{n}p'^{j}_0 p'^{j-1}_1\cdots p'_{j-1}-\sum_{j=2}^{n}p'^{j}_0 p'^{j-1}_1\cdots p'_{j-1}-p'^{n+1}_0 p'^{n}_1\cdots p'_n \\
&= 1-\bar{p'}_n\prod_{l=0}^{n-1}p'^{n-l}_l\,,
\end{aligned}
\tag{3.2.5}
$$

such that

$$
\begin{aligned}
\hat{D}_n(1) &= 1-p'_0+\hat{U}_n(1)+\prod_{l=0}^{n-1}p'^{n-l}_l(1-\bar{p'}_n)\frac{\bar{p'}_n}{1-\bar{p'}_n} \\
&= 1-\bar{p'}_n\prod_{l=0}^{n-1}p'^{n-l}_l+\bar{p'}_n\prod_{l=0}^{n-1}p'^{n-l}_l \\
&= 1
\end{aligned}
$$

The average mean of 1-step walk is defined as

$$
\mu_n=\sum_{j\in\mathbb{Z}}jD_n(j)=\hat{D}'_n(1)\,. \tag{3.2.6}
$$

We will show in next section that

$$
\alpha_{n,c}=\mu_n\,. \tag{3.2.7}
$$

By (3.2.4), we have

$$
\frac{d}{dt}\hat{D}_n(t)=\frac{d}{dt}\hat{U}_n(t)+(\prod_{l=0}^{n-1}p'^{n-l}_l)(1-\bar{p'}_n)\bar{p'}_n\frac{(n+1)t^n-n\bar{p'}_nt^{n+1}}{(1-\bar{p'}_nt)^2}\,, \tag{3.2.8}
$$

where

$$
\frac{d}{dt}\hat{U}_n(t)=\sum_{j=1}^{n}(p'^{j}_0 p'^{j-1}_1\cdots p'_{j-1})(1-p'_0p'_1\cdots p'_{j-1}p'_j)jt^{j-1}\,. \tag{3.2.9}
$$

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Taking $t = 1$ in (3.2.9), we have

$$
\begin{aligned}
\frac{d}{dt}\hat{U}_n(1) &= \sum_{j=1}^{n} (p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1})(1 - p'_0 p'_1 \cdots p'_{j-1} p'_j) j \\
&= \sum_{j=1}^{n} j p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} - \sum_{j=1}^{n} j p'^{j+1}_{0} p'^{j}_{1} \cdots p'^{2}_{j-1} p'_j \\
&= p'_0 + \sum_{j=2}^{n} j p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} - \sum_{j=1}^{n} j p'^{j+1}_{0} p'^{j}_{1} \cdots p'_j \\
&= p'_0 + \sum_{j=1}^{n-1} (j+1) p'^{j+1}_{0} p'^{j}_{1} \cdots p'_j - \sum_{j=1}^{n-1} j p'^{j+1}_{0} p'^{j}_{1} \cdots p'_j - n p'^{n+1}_{0} p'^{n}_{1} \cdots p'_n \\
&= p'_0 + \sum_{j=1}^{n-1} p'^{j+1}_{0} p'^{j}_{1} \cdots p'_j - n p'^{n+1}_{0} p'^{n}_{1} \cdots p'_n \\
&= p'_0 + \sum_{j=2}^{n} p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} - n p'^{n+1}_{0} p'^{n}_{1} \cdots p'_n \\
&= \sum_{j=1}^{n} p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} - n p'^{n+1}_{0} p'^{n}_{1} \cdots p'_n \,.
\end{aligned} \tag{3.2.10}
$$

Therefore from (3.2.6), (3.2.7), (3.2.8), and (3.2.10), we obtain

$$
\begin{aligned}
\alpha_{n,c} &= \sum_{j=1}^{n} p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} - n p'^{n+1}_{0} p'^{n}_{1} \cdots p'_n + \bar{p'}_n \prod_{l=0}^{n-1} p'^{n-l}_{l} \frac{n - n\bar{p'}_n + 1}{1 - \bar{p'}_n} \\
&= \sum_{j=1}^{n} p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} - n p'^{n+1}_{0} p'^{n}_{1} \cdots p'_n + \bar{p'}_n \prod_{l=0}^{n-1} p'^{n-l}_{l} \Big(n + \frac{1}{1 - \bar{p'}_n}\Big) \\
&= \sum_{j=1}^{n} p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} - n\bar{p'}_n \prod_{l=0}^{n-1} p'^{n-l}_{l} + n\bar{p'}_n \prod_{l=0}^{n-1} p'^{n-l}_{l} + \frac{1}{1 - \bar{p'}_n} \bar{p'}_n \prod_{l=0}^{n-1} p'^{n-l}_{l} \\
&= \sum_{j=1}^{n} p'^{j}_{0} p'^{j-1}_{1} \cdots p'_{j-1} + \frac{1}{1 - \bar{p'}_n} \prod_{l=0}^{n} p'^{n+1-l}_{l} \,.
\end{aligned} \tag{3.2.11}
$$

Remark 3.1. *From (3.2.11) we have* $\alpha_{1,c} = p'_0 + \frac{p'^2_0 p'_1}{1 - p'_0 p'_1}$ *and* $\alpha_{0,c} = \frac{p'_0}{1 - p'_0}$ *which are equal to* $\alpha_{1,c}$ *and* $\alpha_{0,c}$ *that are mentioned in chapter 1.*

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3.3 Derivation of σ_n^2

Define the variance of 1-step walk as

$$\begin{aligned} \sigma_n^2 &\equiv \sum_{j\in\mathbb{Z}} (jD_n(j) - \mu_n)^2 \\ &= \sum_{j\in\mathbb{Z}} j^2 D_n(j) - \mu_n^2. \end{aligned} \tag{3.3.1}$$

Since

$$\begin{aligned} \frac{d^2}{dt^2}\hat{D}_n(t) &= \sum_{j=1}^{\infty} j(j-1)D_n(j)t^{j-2} \\ &= \sum_{j=1}^{\infty} j^2 D_n(j)t^{j-2} - \sum_{j=1}^{\infty} jD_n(j)t^{j-2} \end{aligned}$$

and

$$\frac{d^2}{dt^2}\hat{D}_n(1) = \sum_{j=1}^{\infty} j^2 D_n(j) - \alpha_{n,c}\,,$$

a representation of the variance is given by

$$\sigma_n^2 = \frac{d^2}{dt^2}\hat{D}_n(1) + \alpha_{n,c} - \alpha_{n,c}^2\,. \tag{3.3.2}$$

By (3.2.8), we have

$$\begin{aligned} \frac{d^2}{dt^2}\hat{D}_n(t) = \frac{d^2}{dt^2}\hat{U}_n(t) &+ (1-\bar{p'}_n)\bar{p'}_n \\ &\times \prod_{l=0}^{n-1} {p'}_l^{n-l} \frac{n(n+1)t^{n-1}(1-\bar{p'}_n t)^2 + 2m\bar{p'}_n t^n(1-\bar{p'}_n t) + 2\bar{p'}_n t^n}{(1-\bar{p'}_n t)^3}. \end{aligned} \tag{3.3.3}$$

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Note that

$$
\begin{aligned}
\frac{d^2}{dt^2}\hat{U}_n(1) &= \Big[\sum_{j=1}^{n}(p'^{j}_0 p'^{j-1}_1 \cdots p'_{j-1})(1-p'_0 p'_1 \cdots p'_{j-1}p'_j)j(j-1)t^{j-2}\Big]_{t=1} \\
&= \sum_{j=2}^{n}(j^2-j)(p'^{j}_0 p'^{j-1}_1 \cdots p'_{j-1} - p'^{j+1}_0 p'^{j}_1 \cdots p'^{2}_{j-1}p'_j) \\
&= \sum_{j=2}^{n} j^2(p'^{j}_0 p'^{j-1}_1 \cdots p'_{j-1} - p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - \sum_{j=2}^{n} j(p'^{j}_0 p'^{j-1}_1 \cdots p'_{j-1} - p'^{j+1}_0 p'^{j}_1 \cdots p'_j) \\
&= 4p'^2_0 p'_1 + \sum_{j=3}^{n} j^2(p'^{j}_0 p'^{j-1}_1 \cdots p'_{j-1}) - \sum_{j=2}^{n} j^2(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) \\
&\quad - \Big[2p'^2_0 p'_1 + \sum_{j=3}^{n} j(p'^{j}_0 p'^{j-1}_1 \cdots p'_{j-1}) - \sum_{j=2}^{n} j(p'^{j+1}_0 p'^{j}_1 \cdots p'_j)\Big] \\
&= 2p'^2_0 p'_1 + \sum_{j=2}^{n-1}(j+1)^2(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - \sum_{j=2}^{n-1} j^2(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - n^2(p'^{n+1}_0 p'^{n}_1 \cdots p'_n) \\
&\quad - \Big[\sum_{j=2}^{n-1}(j+1)(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - \sum_{j=2}^{n-1} j(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - n(p'^{n+1}_0 p'^{n}_1 \cdots p'_n)\Big] \\
&= 2p'^2_0 p'_1 + \sum_{j=2}^{n-1} 2j(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - (n^2-n)(p'^{n+1}_0 p'^{n}_1 \cdots p'_n) \\
&= \sum_{j=1}^{n-1} 2j(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - (n^2-n)(p'^{n+1}_0 p'^{n}_1 \cdots p'_n).
\end{aligned}
$$

Then we take $t = 1$ into (3.3.3) and obtain

$$
\begin{aligned}
\frac{d^2}{dt^2}\hat{D}_n(1) &= \sum_{j=1}^{n-1} 2j(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) - (n^2-n)(p'^{n+1}_0 p'^{n}_1 \cdots p'_n) \\
&\quad + (p'^{n+1}_0 p'^{n}_1 \cdots p'_n)\frac{n(n+1)(1-\bar{p'}_n)^2 + 2m\bar{p'}_n(1-\bar{p'}_n) + 2\bar{p'}_n}{(1-\bar{p'}_n)^2} \\
&= \sum_{j=1}^{n-1} 2j(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) + (p'^{n+1}_0 p'^{n}_1 \cdots p'_n)\Big[2m + 2m(\frac{1}{1-\bar{p'}_n} - 1) + \frac{2\bar{p'}_n}{(1-\bar{p'}_n)^2}\Big] \\
&= 2\sum_{j=1}^{n-1} j(p'^{j+1}_0 p'^{j}_1 \cdots p'_j) + \frac{2}{1-\bar{p'}_n}(p'^{n+1}_0 p'^{n}_1 \cdots p'_n)\Big(n + \frac{\bar{p'}_n}{1-\bar{p'}_n}\Big).
\end{aligned}
\tag{3.3.4}
$$

Therefore, we find

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$$\sigma_n^2 = 2\sum_{j=1}^{n-1} j({p'}_0^{j+1}{p'}_1^{j}\cdots {p'}_j) + \frac{2}{1-\bar{p'}_n}({p'}_0^{n+1}{p'}_1^{n}\cdots {p'}_n)\left(n+\frac{\bar{p'}_n}{1-\bar{p'}_n}\right) + \alpha_{n,c} - \alpha_{n,c}^2 \quad . \quad (3.3.5)$$

Remark 3.2. *For the long-range Domany-Kinzel model with left direction in [5],*

$$\alpha_{n,c} = \frac{\prod_{j=1}^{n} {p'}_j^{j}}{1-\bar{p'}_n} - \sum_{k=2}^{n}(1 - {p'}_k {p'}_{k+1}^2 \cdots {p'}_n^{n-k+1}) \text{, and}$$

$$\sigma_n^2 = n(n-1) - 2\sum_{k=1}^{n-1} k{p'}_{k+1}\cdots {p'}_n^{n-k} + \frac{2\bar{p'}_n(\prod_{j=1}^{n} {p'}_j^{j})}{(1-\bar{p'}_n)^2} + \alpha_{n,c} - (\alpha_{n,c})^2 \,.$$

3.4 Behavior of $\alpha_{n,c}$ and σ_n as $n \to \infty$

Let $\alpha_c = \lim_{n\to\infty} \alpha_{n,c}$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$.

3.4.1 The case that $p_m \approx_{m\to\infty} \frac{p}{m^s}$

In this subsection we let $p_m \approx_{m\to\infty} \frac{p}{m^s}$ with $p \in (0,1)$ and $s > 1$. We describe the behavior of $\alpha_{n,c}$ as $n \to \infty$ in the following. Recall that

$$\alpha_{n,c} = \sum_{j=1}^{n} {p'}_0^{j}{p'}_1^{j-1}\cdots {p'}_{j-1} + \frac{1}{1-\bar{p'}_n}\prod_{l=0}^{n} {p'}_l^{n+1-l}.$$

Proving $\lim_{n\to\infty} \sum_{j=1}^{n} {p'}_0^{j}{p'}_1^{j-1}\cdots {p'}_{j-1} < \infty$ and $\lim_{n\to\infty}\left[\frac{1}{1-\bar{p'}_n}\prod_{l=0}^{n} {p'}_l^{n+1-l}\right] = 0$ will lead us to $\alpha_c \in (0,\infty)$. Then, first, since ${p'}_j \leq 1 \quad \forall j \in \mathbb{Z}^+$, $\quad {p'}_0^{j}{p'}_1^{j-1}\cdots {p'}_{j-1} \leq {p'}_0^{j}$. We have

$$\lim_{n\to\infty} \sum_{j=1}^{n} {p'}_0^{j}{p'}_1^{j-1}\cdots {p'}_{j-1} \leq \lim_{n\to\infty}\sum_{j=1}^{n} {p'}_0^{j} < \infty. \quad (3.4.1.1)$$

Secondly, we need to prove that $\lim_{n\to\infty} \bar{p'}_n \in (0,1)$ and $\lim_{n\to\infty} \prod_{l=0}^{n} {p'}_l^{n+1-l} = 0$. By definition,

$$\bar{p'}_n = e^{\log({p'}_0 {p'}_1 \cdots {p'}_n)} = e^{\sum_{k=0}^{n} \log(1-p_k)} \approx e^{-\sum_{k=1}^{n} p_k} \approx e^{-\sum_{k=1}^{n} \frac{p}{k^s}}. \quad (3.4.1.2)$$

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Note that

$$\sum_{k=1}^{\infty} \frac{p}{k^s} = \begin{cases} \infty, \text{ if } s \leq 1 \,, \\ \\ \in (0, \infty), \text{ if } s > 1 \,. \end{cases}$$

Hence, we have,

$$\lim_{n\to\infty} \bar{p'}_n = \begin{cases} 0, \text{ if } s \leq 1 \,, \\ \\ \in (0, 1), \text{ if } s > 1 \,. \end{cases} \tag{3.4.1.3}$$

Next, we consider

$$\begin{aligned} \prod_{l=0}^{n} {p'}_l^{n+1-l} &= e^{\log({p'}_0^{n+1}{p'}_1^{n}\cdots {p'}_n)} \\ &= e^{\sum_{k=0}^{n}(n+1-k)\log(1-p_k)} \\ &\approx e^{-\sum_{k=1}^{n}(n+1-k)p_k} \\ &\approx e^{-\sum_{k=1}^{n}(n+1-k)\frac{p}{k^s}}. \end{aligned} \tag{3.4.1.4}$$

To show that $\lim_{n\to\infty} \prod_{l=0}^{n} {p'}_l^{n+1-l} = 0$, it is sufficient to show that $\lim_{n\to\infty} \sum_{k=1}^{n}(n+1-k)\frac{p}{k^s} = \infty$. We seperate it into two cases. In the first case that $s \geq 2$, we derive

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{n}(n+1-k)\frac{p}{k^s} &= \lim_{n\to\infty} \sum_{k=1}^{n} \left[(n+1)\frac{p}{k^s} - \frac{p}{k^{s-1}}\right] \\ &= \lim_{n\to\infty} \left[(n+1)p\sum_{k=1}^{n}\frac{1}{k^s} - p\sum_{k=1}^{n}\frac{1}{k^{s-1}}\right] \\ &= \infty. \end{aligned} \tag{3.4.1.5}$$

In the second case that $s \in (1, 2)$, first we consider

$$\sum_{k=1}^{n}(n+1-k)\frac{p}{k^s} \approx \sum_{k=1}^{n}(n+1-k)\frac{1}{k^s} = \sum_{j=1}^{n}\frac{j}{(n+1-j)^s}.$$

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Note that

$$
\begin{aligned}
\sum_{j=1}^{n+1} \frac{j}{(n+1-j)^s} &= \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{j}{(n+1-j)^s} + \sum_{j=\left[\frac{n+1}{2}\right]+1}^{n+1} \frac{j}{(n+1-j)^s} \\
&\geq \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{j}{(n+1-j)^s} \\
&\geq \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{j}{n^s} \\
&= \frac{1}{n^s} \frac{\left[\frac{n+1}{2}\right]\left(\left[\frac{n+1}{2}\right]+1\right)}{2} \\
&\approx n^{2-s} \to \infty \text{ as } n \to \infty.
\end{aligned}
\tag{3.4.1.6}
$$

Hence, by (3.4.1.5) and (3.4.1.6), we have,

$$\lim_{n\to\infty} \prod_{l=0}^{n} {p'}_l^{n+1-l} = 0 \text{ , if } s > 1. \tag{3.4.1.7}$$

Finally, by (3.4.1.1), (3.4.1.3), and(3.4.1.7), we conclude that

$$\alpha_c \in (0,\infty) \text{ , when } s > 1. \tag{3.4.1.8}$$

Next, we describe the behavior of σ_n as $n \to \infty$ in the following. Recall that

$$\sigma_n^2 = 2\sum_{j=1}^{n-1} j({p'}_0^{j+1}{p'}_1^{j}\cdots {p'}_j) + \frac{2}{1-\bar{p'}_n}({p'}_0^{n+1}{p'}_1^{n}\cdots {p'}_n)\left(n + \frac{\bar{p'}_n}{1-\bar{p'}_n}\right) + \alpha_{n,c} - \alpha_{n,c}^2.$$

To say that $\lim_{n\to\infty} \sigma_n^2 \in (0,\infty)$, it is sufficient to show that $\lim_{n\to\infty} \sum_{j=1}^{n-1} j({p'}_0^{j+1}{p'}_1^{j}\cdots {p'}_j) \in (0,\infty)$ and $\lim_{n\to\infty} n({p'}_0^{n+1}{p'}_1^{n}\cdots {p'}_n) = 0$. First, we rewrite

$$
\begin{aligned}
\sum_{j=1}^{n-1} j({p'}_0^{j+1}{p'}_1^{j}\cdots {p'}_j) &= \sum_{j=1}^{n-1} j e^{\sum_{k=0}^{j}(j+1-k)\log(1-p_k)} \\
&\approx \sum_{j=1}^{n-1} j e^{-\sum_{k=1}^{j}(j+1-k)\frac{1}{k^s}}.
\end{aligned}
\tag{3.4.1.9}
$$

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Similar to the proof of α_c, we seperate it into three cases. In the case that $s > 2$,

$$\begin{aligned}\sum_{k=1}^{j}(j+1-k)\frac{1}{k^s} &= (j+1)\sum_{k=1}^{j}\frac{1}{k^s} - \sum_{k=1}^{j}\frac{1}{k^{s-1}} \\ &\approx j+1.\end{aligned} \tag{3.4.1.10}$$

In the case that $s = 2$,

$$\begin{aligned}\sum_{k=1}^{j}(j+1-k)\frac{1}{k^s} &= (j+1)\sum_{k=1}^{j}\frac{1}{k^2} - \sum_{k=1}^{j}\frac{1}{k} \\ &\approx j+1-\log j.\end{aligned} \tag{3.4.1.11}$$

In the case that $s \in (1,2)$,

$$\begin{aligned}\sum_{k=1}^{j}(j+1-k)\frac{1}{k^s} &= \sum_{l=1}^{j}\frac{l}{(j+1-l)^s} \\ &\approx \sum_{l=1}^{j}\frac{l}{j^s} \\ &\approx j^{2-s}.\end{aligned} \tag{3.4.1.12}$$

Put the results of $\sum_{k=1}^{j}(j+1-k)\frac{1}{k^s}$ back in (3.4.1.9), we get

$$\sum_{j=1}^{n-1} j({p'}_0^{j+1}{p'}_1^{j}\cdots {p'}_j) \approx_{n\to\infty} \begin{cases} \sum_{j=1}^{n-1}\frac{j}{e^{j+1}}, \text{ if } s>2\,, \\ \sum_{j=1}^{n-1}\frac{je^{\log j}}{e^{j+1}}, \text{ if } s=2\,, \\ \sum_{j=1}^{n-1}\frac{j}{e^{j^{2-s}}}, \text{ if } s\in(1,2)\,. \end{cases}$$

Hence, we conclude that, as $n \to \infty$,

$$\lim_{n\to\infty}\sum_{j=1}^{n-1} j({p'}_0^{j+1}{p'}_1^{j}\cdots {p'}_j) < \infty \text{ , if } s>1. \tag{3.4.1.13}$$

In the case that $s > 1$, since $\sum_{j=1}^{n-1} j({p'}_0^{j+1}{p'}_1^{j}\cdots {p'}_j) < \infty$ by (3.4.1.13), using the fact that if

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$\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$,

$$\lim_{n\to\infty} n{p'}_0^{n+1}{p'}_1^{n}\cdots{p'}_n = 0. \tag{3.4.1.14}$$

Thus, by (3.4.1.3), (3.4.1.13), and (3.4.1.14), we conclude that

$$\sigma^2 \in (0,\infty)\text{ , when } s > 1. \tag{3.4.1.15}$$

Remark 3.3. *For the long-range Domany-Kinzel model with left direction in the case that $p_m \approx_{m\to\infty} \frac{p}{m^s}$ in [5], $\alpha_c \in (-\infty, 1)$ for $s > 3$ and $\sigma^2 \in (0,\infty)$ for $s > 4$.*

3.4.2 The case that $p_m = \frac{e^{-\lambda}\lambda^m}{m!}$

In this subsection we let $p_m = \frac{e^{-\lambda}\lambda^m}{m!}$ with $\lambda > 0$. We use the same argument as 3.4.1 as following. Recall that

$$\alpha_{n,c} = \sum_{j=1}^{n} {p'}_0^{j}{p'}_1^{j-1}\cdots{p'}_{j-1} + \frac{1}{1-\bar{p'}_n}\prod_{l=0}^{n}{p'}_l^{n+1-l}.$$

Proving $\lim_{n\to\infty}\sum_{j=1}^{n} {p'}_0^{j}{p'}_1^{j-1}\cdots{p'}_{j-1} < \infty$ and $\lim_{n\to\infty}\left[\frac{1}{1-\bar{p'}_n}\prod_{l=0}^{n}{p'}_l^{n+1-l}\right] = 0$ will lead us to $\alpha_c \in (0,\infty)$. Then, first, since ${p'}_j \le 1 \quad \forall j \in \mathbb{Z}^+$, ${p'}_0^{j}{p'}_1^{j-1}\cdots{p'}_{j-1} \le {p'}_0^{j}$.
We have

$$\lim_{n\to\infty}\sum_{j=1}^{n} {p'}_0^{j}{p'}_1^{j-1}\cdots{p'}_{j-1} \le \lim_{n\to\infty}\sum_{j=1}^{n}{p'}_0^{j} < \infty. \tag{3.4.2.1}$$

Secondly, we have

$$\lim_{n\to\infty}\bar{p'}_n \approx \lim_{n\to\infty} e^{-\sum_{k=0}^{n} p_k} \approx \lim_{n\to\infty} e^{-\sum_{k=0}^{n}\frac{\lambda^k e^{-\lambda}}{k!}} = \lim_{n\to\infty} e^{-1}. \tag{3.4.2.2}$$

Thirdly, we consider

$$\prod_{l=0}^{n}{p'}_l^{n+1-l} \approx e^{-\sum_{k=0}^{n}(n+1-k)p_k} \approx e^{-\sum_{k=0}^{n}(n+1-k)\frac{\lambda^k e^{-\lambda}}{k!}}. \tag{3.4.2.3}$$

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Note that $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$ conveges for all $\lambda > 0$. Then, we know that

$$\begin{aligned} \sum_{k=0}^{n}(n+1-k)\frac{\lambda^k e^{-\lambda}}{k!} &= e^{-\lambda}\Big[(n+1)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} - \sum_{k=0}^{\infty}\frac{\lambda^k}{(k-1)!}\Big] \\ &\approx n+1 \to \infty \text{ as } n \to \infty. \end{aligned} \tag{3.4.2.4}$$

Hence, we have

$$\lim_{n\to\infty}\prod_{l=0}^{n} p'^{\,n+1-l}_{l} = 0. \tag{3.4.2.5}$$

Finally, by (3.4.2.1), (3.4.2.2), and(3.4.2.5), we conclude that

$$\alpha_c \in (0,\infty)\text{ , when } p_m = \frac{e^{-\lambda}\lambda^m}{m!}, \quad \forall\lambda > 0. \tag{3.4.2.6}$$

Next, we describe the behavior of σ_n as $n \to \infty$ in the following. We rewrite

$$\begin{aligned} \sum_{j=1}^{n-1} j(p'^{\,j+1}_{0} p'^{\,j}_{1} \cdots p'_{j}) &= \sum_{j=1}^{n-1} j e^{\sum_{k=0}^{j}(j+1-k)\log(1-p_k)} \\ &\approx \sum_{j=1}^{n-1} j e^{-\sum_{k=1}^{j}(j+1-k)\frac{\lambda^k e^{-\lambda}}{k!}}. \end{aligned} \tag{3.4.2.7}$$

Similar to (3.4.2.4), $\sum_{k=1}^{j}(j+1-k)\frac{\lambda^k e^{-\lambda}}{k!} \approx j+1$.

Hence, we have,

$$\lim_{n\to\infty}\sum_{j=1}^{n-1} j(p'^{\,j+1}_{0} p'^{\,j}_{1} \cdots p'_{j}) \approx \lim_{n\to\infty}\sum_{j=1}^{n-1}\frac{j}{e^{j+1}} < \infty. \tag{3.4.2.8}$$

Since $\lim_{n\to\infty}\sum_{j=1}^{n-1} j(p'^{\,j+1}_{0} p'^{\,j}_{1} \cdots p'_{j}) < \infty$, $\lim_{n\to\infty} n p'^{\,n+1}_{0} p'^{\,n}_{1} \cdots p'_{n} = 0$. Thus, by (3.4.2.2) and (3.4.2.8), we conclude that

$$\sigma^2 \in (0,\infty)\text{ , when } p_m = \frac{e^{-\lambda}\lambda^m}{m!}, \quad \forall\lambda > 0. \tag{3.4.2.9}$$

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Chapter 4

The proof of main theorem

4.1 Proof of Theorem 2.1

Let $n \in \mathbb{N}$ be fixed in this section. The following proof is based on section 4 in [5]. Since the expression of $\alpha_{n,c}$ has been obtained in section 3.2, we shall go into the behavior of $\tau_n(M_\alpha, M)$ when α is close to $\alpha_{n,c}$. Define a M-step random walk $S_{n,M}$ with the distribution D_n for each step. To avoid confusion with the probability $\mathbb{P}$ defined in chapter 1, we define the probability Prob_n such that $\text{Prob}_n(S_{n,M} = j) = C_{n,M}(j)$ with $j \in \mathbb{Z}$ and $\text{Prob}_n(S_{n,0} = j) = \delta_{0,j}$, the Kronecker delta. The expectation for Prob_n is denoted by Exp_n. Let $X_1, X_2, \cdots, X_M$ be i.i.d. random variables with distribution D_n. Then, by the result in section 3.2 and 3.3, the random variables have finite mean ($\alpha_{n,c}$), variance (σ_n^2), and third moment. Since each step of $S_{n,M}$ is independent, we obtain that $S_{n,M} = X_1 + X_2 + \cdots + X_M$. By the law of large number, we have

$$\frac{S_{n,M}}{M} \to \alpha_{n,c} \quad \text{a.s. when } M \to \infty. \tag{4.1.1}$$

Notice that, since $S_{n,M} = X_1 + X_2 + \cdots + X_M$, the mean and variance of $S_{n,M}$ are given by $\alpha_{n,c}M$ and $M\sigma_n^2$, respectively. Berry-Esseen theorem (c.f. [9]) asserts that

$$\left|\text{Prob}_n\Big(\frac{S_{n,M} - \alpha_{n,c}M}{\sqrt{M\sigma_n^2}} \le \frac{M(\alpha - \alpha_{n,c})}{\sqrt{M\sigma_n^2}}\Big) - \int_{-\infty}^{\frac{M(\alpha-\alpha_{n,c})}{\sqrt{M\sigma_n^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right| \le O\Big(\frac{\sum_{j\in\mathbb{Z}} |j|^3 D_n(j)}{\sqrt{M}\sigma_n^3}\Big). \tag{4.1.2}$$

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With the definition of M_α given in the introduction, we have

$$\text{Prob}_n(S_{n,M} \le \alpha M - 1) \le \tau_n(M_\alpha, M) = \text{Prob}_n(S_{n,M} \le M_\alpha) \le \text{Prob}_n(S_{n,M} \le \alpha M). \tag{4.1.3}$$

Setting $\alpha = \alpha_{n,c}$ and using $\sum_{j\in\mathbb{Z}} |j|^3 D_n(j) < \infty$, we obtain

$$\begin{aligned}\tau_n(M_{\alpha_{n,c},M}) &= \text{Prob}_n(S_{n,M} \le \alpha_{n,c}M) = \text{Prob}_n\Big(\frac{S_{n,M} - \alpha_{n,c}M}{\sqrt{M}\sigma_n} \le 0\Big) \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O(\frac{1}{\sqrt{M}}) = \frac{1}{2} + O(\frac{1}{\sqrt{M}}),\end{aligned} \tag{4.1.4}$$

which gives (2.2).

Note that the Chernoff inequality states that

$$P(X \ge a) = P(e^{rX} \ge e^{ra}) \le \inf_{r>0} \frac{E(e^{rX})}{e^{ra}},$$

where X is the sum of n independent random variables and $a \in \mathbb{R}$.
We consider a general $\alpha \neq \alpha_{n,c}$ in the rest part of this section. When $\alpha < \alpha_{n,c}$, we let $\eta = -\log r > 0$ and deal $Prob_n(S_{n,M} \le M_\alpha)$ with Chernoff inequality to obtain

$$\begin{aligned}\text{Prob}_n(S_{n,M} \le M_\alpha) &\le \text{Prob}_n(S_{n,M} \le \alpha M) = \text{Prob}_n(e^{-\eta S_{n,M}} \ge e^{-\eta\alpha M}) \\ &\le \inf_{\eta>0} \frac{\text{Exp}_n(e^{-\eta S_{n,M}})}{e^{-\eta\alpha M}} = \inf_{r\in(0,1)} \frac{\hat{S}_{n,M}(r)}{r^{\alpha M}} \\ &= \inf_{r\in(0,1)} \frac{\hat{D}_n(r)^M}{r^{\alpha M}} \le e^{-MI_n(\alpha)},\end{aligned} \tag{4.1.5}$$

where

$$I_n(\alpha) = \sup_{r>0} \big\{\alpha \log r - \log \hat{D}_n(r)\big\} := \alpha \log r_\alpha - \log \hat{D}_n(r_\alpha). \tag{4.1.6}$$

With similar argument, when $\alpha > \alpha_{n,c}$, we let $\eta = \log r > 0$ to obtain

$$\begin{aligned}\text{Prob}_n(S_{n,M} > M_\alpha) &= \inf_{\eta>0} \text{Prob}_n(e^{\eta S_{n,M}} > e^{\eta\alpha M - \eta c}) \text{, where } c \in [0,1) \\ &\approx_{M\to\infty} \inf_{\eta>0} \text{Prob}_n(e^{\eta S_{n,M}} > e^{\eta\alpha M}) \\ &\le \inf_{\eta>0} \frac{\text{Exp}_n(e^{\eta S_{n,M}})}{e^{\eta\alpha M}} \le e^{-MI_n(\alpha)}.\end{aligned} \tag{4.1.7}$$

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By (4.1.6), we have

$$\frac{\alpha}{r_\alpha} = \frac{\hat{D}'_n(r_\alpha)}{\hat{D}_n(r_\alpha)}, \tag{4.1.8}$$

such that

$$I'_n(\alpha) = \log r_\alpha - \Big(\frac{\hat{D}'_n(r_\alpha)}{\hat{D}_n(r_\alpha)} - \frac{\alpha}{r_\alpha}\Big) r'_\alpha = \log r_\alpha. \tag{4.1.9}$$

As $\hat{D}_n(1) = 1$ and $\hat{D}'_n(1) = \alpha_{n,c}$, replacing $r_\alpha = 1$ in (4.1.8) we have $r_{\alpha_{n,c}} = \hat{D}_n(r_{\alpha_{n,c}}) = 1$ and hence $I_n(\alpha_{n,c}) = I'_n(\alpha_{n,c}) = 0$. Moreover, using the generating function of D and (4.1.8), we get

$$\alpha = \frac{r_\alpha \hat{D}'_n(r_\alpha)}{\hat{D}_n(r_\alpha)} = \frac{\sum_{m\in\mathbb{Z}} m D_n(m) r_\alpha^m}{\sum_{m\in\mathbb{Z}} D_n(m) r_\alpha^m}. \tag{4.1.10}$$

Let us take derivative on both sides of (4.1.8) with respect to α. Then we obtain

$$\frac{1}{r_\alpha} - \frac{\alpha r'_\alpha}{r_\alpha^2} = \Big(\frac{\hat{D}''_n(r_\alpha)}{\hat{D}(r_\alpha)} - \big(\frac{\hat{D}'_n(r_\alpha)}{\hat{D}_n(r_\alpha)}\big)^2\Big) r'_\alpha.$$

Again, taking derivative on (4.1.9), we obtain, for all $\alpha \in \mathbb{R}$,

$$I''_n(\alpha) = \frac{r'_\alpha}{r_\alpha} = \frac{1}{V_n(\alpha)}, \quad \text{where } V_n(\alpha) = \frac{\sum_{m\in\mathbb{Z}} m^2 D_n(m) r_\alpha^m}{\hat{D}_n(r_\alpha)} - \alpha^2 \in (0, \infty). \tag{4.1.11}$$

Hence, we can conclude that $I_n(\alpha)$ is strictly convex $\forall \alpha \in \mathbb{R}$ and it has local minimum at $\alpha_{n,c}$ with $I_n(\alpha_{n,c}) = 0$.

Since r_α is a continuous function with respect to α, we acquire $|r_\alpha - r_{\alpha_{n,c}}| \in (0,1)$ as α is near $\alpha_{n,c}$. From (4.1.9) we have

$$I'_n(\alpha) = \log r_\alpha = \log\big(r_{\alpha_{n,c}} + (r_\alpha - r_{\alpha_{n,c}})\big) = (r_\alpha - r_{\alpha_{n,c}}) + O(1)(r_\alpha - r_{\alpha_{n,c}})^2.$$

Applying mean value theorem and the fact that $I_n(\alpha)$ is strictly convex, it follows that $I'_n(\alpha) \approx I''_n(\alpha_{n,c})(\alpha - \alpha_{n,c})$. Therefore, $r_\alpha - r_{\alpha_{n,c}} \approx_{\alpha\to\alpha_{n,c}} \alpha - \alpha_{n,c}$, so that

$$I_n(\alpha) = \int_{\alpha_{n,c}}^{\alpha} I'_n(u) du \approx_{\alpha\to\alpha_{n,c}} \int_{\alpha_{n,c}}^{\alpha} (r_u - r_{\alpha_{n,c}}) du \approx_{\alpha\to\alpha_{n,c}} \int_{\alpha_{n,c}}^{\alpha} (u - \alpha_{n,c}) du = \frac{(\alpha - \alpha_{n,c})^2}{2}. \tag{4.1.12}$$

Thus, the proof of Theorem 2.1 is completed.

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4.2 Proof of Theorem 2.2

In the case of $n \to \infty$, we consider $p_m \in [0,1)$ for $m \in \mathbb{Z}_+$ and $p_m \approx_{m\to\infty} \frac{p}{m^s}$ with $p \in (0,1)$ and $s > 1$. By the result of section 3.4.1, $\alpha_c \in (0,\infty)$ for $s > 1$ and $\sigma^2 \in (0,\infty)$ for $s > 1$.

Denote the probability measure Prob such that $\text{Prob}(S_M = j) = C_M(j)$ with $j \in \mathbb{Z}$ and $\text{Prob}(S_0 = j) = C_0(j) = \delta_{0,j}$ where $C_M(k) = \sum_{j\in\mathbb{Z}} C_{M-1}(k-j)D(j)$ for $M \geq 1$. The expectation with respect to Prob is denoted by Exp. To use Berry-Esseen theorem, we need to show that $\sum_{k=1}^{\infty} k^3 D(k) < \infty$. Recall that, in (3.1.3) with $n \to \infty$, $D(j) = (p'^j_0 p'^{j-1}_1 \cdots p'_{j-1})(1 - p'_0 p'_1 \cdots p'_j)$ for $j \geq 1$.

$$\begin{aligned}
\sum_{j=1}^{\infty} j^3 D(j) &= \sum_{j=1}^{\infty} j^3 (p'^j_0 p'^{j-1}_1 \cdots p'_{j-1})(1 - p'_0 p'_1 \cdots p'_j) \\
&= \sum_{j=1}^{\infty} \left(j^3 p'^j_0 p'^{j-1}_1 \cdots p'_{j-1} - j^3 p'^{j+1}_0 p'^j_1 \cdots p'_j \right) \\
&= p'_0 + \sum_{j=2}^{\infty} j^3 p'^j_0 p'^{j-1}_1 \cdots p'_{j-1} - \sum_{j=1}^{\infty} j^3 p'^{j+1}_0 p'^j_1 \cdots p'_j \\
&= p'_0 + \sum_{j=1}^{\infty} (j+1)^3 p'^{j+1}_0 p'^j_1 \cdots p'_j - \sum_{j=1}^{\infty} j^3 p'^{j+1}_0 p'^j_1 \cdots p'_j \\
&= p'_0 + \sum_{j=1}^{\infty} (3j^2 + 3j + 1)(p'^{j+1}_0 p'^j_1 \cdots p'_j) \\
&\leq p'_0 + \sum_{j=1}^{\infty} (3j^2 + 3j + 1) p'^{j+1}_0 < \infty
\end{aligned} \tag{4.2.1}$$

Hence, we can use Berry-Esseen theorem and, with similar argument as (4.1.2) to (4.1.4), we have

$$\tau(M_{\alpha_c}, M) = \frac{1}{2} + O(\frac{1}{\sqrt{M}}), \quad s > 1, \tag{4.2.2}$$

which gives (2.10).

We consider a general $\alpha \neq \alpha_c$ in the rest part of this section. To use similar method in section 4.1 to generate the rate function, we need to check the range of r such that $\hat{D}(r)$ is convergent. Note that, with similar argument as (3.4.1.2),

$$p'_0 p'_1 \cdots p'_j \approx e^{-\sum_{k=1}^{j} \frac{p}{k^s}} \approx e^{-\int_1^j \frac{p}{k^s} dk} \approx e^{\frac{1}{j^{s-1}}} \tag{4.2.3}$$

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and, with similar arguments as (3.4.1.5) and (3.4.1.6),

$$p'^{j}_{0}p'^{j-1}_{1}\cdots p'_{j-1}=\begin{cases} e^{-cj}, c\in\mathbb{R}, & \text{if } s\geq 2,\\ e^{-j^{2-s}}, & \text{if } s\in(1,2).\end{cases} \tag{4.2.4}$$

Hence,

$$\hat{D}(r)=\sum_{j=0}^{\infty}D(j)r^{j}\approx\begin{cases} \sum_{j=1}^{\infty}\frac{r^{j}}{e^{cj}}, & \text{if } s\geq 2,\\ \sum_{j=1}^{\infty}\frac{r^{j}}{e^{j^{2-s}}}, & \text{if } s\in(1,2)\end{cases} \tag{4.2.5}$$

is convergent when $r\in(0,1+\epsilon)$ for some $\epsilon>0$.

Thus, we can use similar method in section 4.1. First, we consider the case that $\alpha<\alpha_c$. By using the same argument as (4.1.5), we obtain

$$\text{Prob}(S_M\leq M_\alpha)\leq\inf_{r\in(0,1)}\frac{\text{Exp}(e^{-\eta S_M})}{e^{-\eta\alpha M}}\leq e^{-MI(\alpha)}, \tag{4.2.6}$$

where

$$I(\alpha)=\sup_{r\in(0,1)}\left\{\alpha\log r-\log\hat{D}(r)\right\}:=\alpha\log r_\alpha-\log\hat{D}(r_\alpha). \tag{4.2.7}$$

Secondly, we consider the case that $\alpha>\alpha_c$. By using the same argument as (4.1.7), we obtain

$$\text{Prob}(S_M>M_\alpha)\leq\inf_{r\in(1,1+\epsilon)}\frac{\text{Exp}(e^{\eta S_M})}{e^{\eta\alpha M}}\leq e^{-MI(\alpha)}, \tag{4.2.8}$$

According to (4.2.7), we have

$$\frac{\alpha}{r_\alpha}=\frac{\hat{D}'(r_\alpha)}{\hat{D}(r_\alpha)}\text{, and } I'(\alpha)=\log r_\alpha. \tag{4.2.9}$$

As $\hat{D}(1)=1$ and $\hat{D}'(1)=\alpha_c$, setting $r_\alpha=1$ in (4.2.5) leads to $r_{\alpha_c}=\hat{D}(r_{\alpha_c})=1$ and $I(\alpha_c)=I'(\alpha_c)=0$. Furthermore, using similar argument from (4.1.10) to (4.1.12), we get

$$I(\alpha)\approx_{\alpha\to\alpha_c}(\alpha-\alpha_c)^2 \quad \text{for } s>1,$$

which completes the proof of Theorem 2.2.

Remark 4.1. *To prove Theorem 2.3, we use similar method as the proof of Theorem 2.2. Here*

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we omit the proof.

4.3 Proof of Theorem 2.4

We consider the same definition of p_m in section 4.2 and let $\alpha_M^- = \alpha_c - M^{-\frac{\rho}{2}}\ell_M, \alpha_M^+ = \alpha_c + M^{-\frac{\rho}{2}}\ell_M$, where $\rho \in (0, \infty)$ and $\{\ell_M\}_{M=1}^{\infty}$ is a positive slowly varying sequence. Note that, by (4.2.1), we can use Berry-Esseen theorem to analysis $\tau(M_{\alpha_M^-}, M)$ and $\tau(M_{\alpha_M^+}, M)$. Hence, we have

$$\begin{aligned} \tau(M_{\alpha_M^-}, M) &= \int_{-\infty}^{\frac{M(-M^{-\frac{\rho}{2}}\ell_M)}{\sigma\sqrt{M}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big), \\ 1 - \tau(M_{\alpha_M^+}, M) &= \int_{-\infty}^{\frac{M(-M^{-\frac{\rho}{2}}\ell_M)}{\sigma\sqrt{M}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big). \end{aligned} \tag{4.3.1}$$

Here we analyse $\tau(M_{\alpha_M^-}, M)$ and $1 - \tau(M_{\alpha_M^+}, M)$ in large M limit for several cases in the following.

First, we consider $\rho > 1$. Recall that, by Tylor's expansion,

$$\begin{aligned} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du &\simeq \int_a^b \frac{1}{\sqrt{2\pi}} \big(1 - \frac{u^2}{2} + o(u^4)\big) du \\ &\simeq (u - \frac{u^3}{6})\Big|_a^b \\ &\approx (b-a)\Big(1 + O\big((b-a)^2\big)\Big), \text{ where } a, b \in \mathbb{R}, a < b, \text{ and } b - a \text{ is small.} \end{aligned} \tag{4.3.2}$$

We simplify the upper limit of (4.3.1) $\frac{M(-M^{-\frac{\rho}{2}}\ell_M)}{\sigma\sqrt{M}} = -\frac{\ell_M}{\sigma M^{\frac{\rho-1}{2}}}$. Since $\frac{\rho-1}{2} > 0$ in this case, $\frac{M(-M^{-\frac{\rho}{2}}\ell_M)}{\sigma\sqrt{M}} \to 0$ as $M \to \infty$. Hence, we obtain

$$\begin{aligned} \tau(M_{\alpha_M^-}, M) &= \int_{-\infty}^{-\frac{\ell_M}{\sigma M^{\frac{\rho-1}{2}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big) \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du - \int_0^{\frac{\ell_M}{\sigma M^{\frac{\rho-1}{2}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big) \\ &\approx \frac{1}{2} - O\big(\frac{\ell_M}{\sigma M^{\frac{\rho-1}{2}}}\big) + O\big(\frac{1}{\sqrt{M}}\big) \\ &= \begin{cases} \frac{1}{2} + O\big(\frac{1}{\sqrt{M}}\big) & \text{if } \rho > 2, \\ \frac{1}{2} + O\big(\frac{\ell_M}{M^{\frac{\rho-1}{2}}}\big) & \text{if } \rho \in (1, 2]. \end{cases} \end{aligned} \tag{4.3.3}$$

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Secondly, we consider $\rho = 1$. Recall that

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}\big(1 + O(x^{-2})\big) \text{ for large } x. \tag{4.3.4}$$

We simplify the upper limit of (4.3.1) $\frac{M(-M^{-\frac{\rho}{2}}\ell_M)}{\sigma\sqrt{M}} = -\frac{\ell_M}{\sigma}$ in this case. Consider that $\lim_{M\to\infty} \ell_M = L$, $L \in [0, \infty)$. Then we obtain

$$\begin{aligned}
\tau(M_{\alpha_M^-}, M) &= \int_{\frac{\ell_M}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big) \\
&= \int_{\frac{L}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + \int_{\frac{\ell_M}{\sigma}}^{\frac{L}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big) \\
&= \Psi(\frac{L}{\sigma}) + O(1) \max\big\{|\ell_M - L|, \frac{1}{\sqrt{M}}\big\}.
\end{aligned} \tag{4.3.5}$$

Next we consider that $\lim_{M\to\infty} \ell_M = \infty$. Then we obtain, by using (4.3.4),

$$\begin{aligned}
\tau(M_{\alpha_M^-}, M) &= \int_{\frac{\ell_M}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big) \\
&= O(1) \max\big\{\frac{\sigma}{\ell_M} e^{-\frac{\ell_M^2}{2\sigma^2}}, \frac{1}{\sqrt{M}}\big\}.
\end{aligned} \tag{4.3.6}$$

In this case, we need to consider another situation which is the result by the large deviation argument. Recall that $\tau(M_\alpha, M) \leq e^{-MI(\alpha)}$ and $I(\alpha) \approx_{\alpha\to\alpha_c} (\alpha - \alpha_c)^2$. We simplify that $\alpha_M^- - \alpha_c = -M^{-\frac{1}{2}}\ell_M$ in this case. Then we obtain

$$\begin{aligned}
\tau(M_{\alpha_M^-}, M) &\leq e^{-MI(\alpha_M^-)} \\
&\approx O(1) e^{-tM(\alpha_M^- - \alpha_c)^2}, \text{ where } t \in (0, \infty) \\
&= O(1) e^{-t\ell_M^2}.
\end{aligned} \tag{4.3.7}$$

At last, we consider $\rho \in (0, 1)$. We simplify that $\frac{M(-M^{-\frac{\rho}{2}}\ell_M)}{\sigma\sqrt{M}} = -\frac{M^{\frac{1-\rho}{2}}\ell_M}{\sigma} \to -\infty$ as $M \to \infty$ in this case. Then we have, by (4.3.1) and (4.3.4),

$$\begin{aligned}
\tau(M_{\alpha_M^-}, M) &= \int_{\frac{M^{\frac{1-\rho}{2}}\ell_M}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + O\big(\frac{1}{\sqrt{M}}\big) \\
&= O(1) \max\big\{\frac{\sigma}{M^{\frac{1-\rho}{2}}\ell_M} e^{-\frac{M^{1-\rho}\ell_M^2}{2\sigma^2}}, \frac{1}{\sqrt{M}}\big\}
\end{aligned} \tag{4.3.8}$$

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and by using large deviation argument we obtain

$$\begin{aligned}\tau(M_{\alpha_M^-}, M) &\leq e^{-MI(\alpha_M^-)} \\ &\approx O(1)e^{-tM(\alpha_M^- - \alpha_c)^2} \text{ where } t \in (0, \infty) \\ &= O(1)e^{-tM^{1-\rho}\ell_M^2}.\end{aligned} \tag{4.3.9}$$

By the same method, we can get the corresponding result for $\alpha = \alpha_M^+$. Thus, the proof of Theorem 2.4 is completed.

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