3 Blow-up Phenomena

Definition 3.1 *A function $g : \mathbb{R} \to \mathbb{R}$ blows up and has a blow-up rate q means that there is a finite number T^* such that the following is valid*

$$\lim_{t \to T^*} g(t)^{-1} = 0 \tag{3.1}$$

and there exists a nonzero $\beta \in \mathbb{R}$ with

$$\lim_{t \to T^*} (T^* - t)^q g(t) = \beta, \tag{3.2}$$

in this case β is called the blow-up constant of g.

Theorem 3.2 *Suppose that u is the classical solution of* (2.1)*. If $u_1 = 0$ then u is constant and $u(t) = u_0$.*

Proof. Together Theorem 2.1 and Theorem 2.2, the solution of differential equation (2.1) on $[0, T)$ is unique, so $u(t) = u_0$ is the unique solution of (2.1) on $[0, T^-]$. Next we consider the following differential equation.

$$\begin{cases} v''(t) = v'(t)^q (c_1 + c_2 v(t)^p), \\ v(0) = u(T^-), v'(0) = u'(T^-). \end{cases}$$

Similarly, $v(t) = u_0$ is the unique solution of the last differential equation on $[0, T^-]$. Let

$$U(t) = \begin{cases} u(t) & \text{if } t \in [0, T^-), \\ v(t - T^-) & \text{if } t \in [T^-, 2T^-], \end{cases}$$

then $U(t) = u_0$ is the unique solution of nonlinear equation (2.1) for $t \in [0, 2T^-]$. Such a way can always be continued forever. Thus $u(t) = u_0$ is the unique solution for $t \in [0, \infty)$. ❑

As $u_1 = 0$, the solution of problem (2.1) u must be constant. Now we consider the situation $u_1 \neq 0$ for the differential equation (2.1),

$$\begin{cases} u''(t) = u'(t)^q (c_1 + c_2 u(t)^p), \\ u(0) = u_0 \neq 0, u'(0) = u_1 \neq 0. \end{cases}$$

For $u_1 \neq 0$ and $t \in [0, T^*)$, where $T^* = \inf\{t > 0 : u'(t) = 0\}$, we have

$$\begin{cases} u'(t)^{2-q} = (2-q)(c_1 u(t) + \dfrac{c_2}{p+1} u(t)^{p+1} + E(0)) \quad \text{if } q \neq 2, \\ \\ E(0) = \dfrac{u_1^{2-q}}{2-q} - (c_1 u_0 + \dfrac{c_2}{p+1} u_0^{p+1}) \end{cases} \tag{3.3}$$

and

$$\begin{cases} \ln |\, u'(t) \,| = (c_1 u(t) + \dfrac{c_2}{p+1} u(t)^{p+1} + E_1(0)) \quad \text{if } q = 2, \\ \\ E_1(0) = \ln |\, u_1 \,| - (c_1 u_0 + \dfrac{c_2}{p+1} u_0^{p+1}). \end{cases} \tag{3.4}$$

Thus we have the relations between $u(t)$ and $u'(t)$.

For a given function u in this work we use the following abbreviations

$$a(t) = u(t)^2, \; J(t) = a(t)^{-m}, \; m = \frac{1}{2}(\frac{1}{2-q} - 1).$$

Lemma 3.3 *Suppose that $f \in C^1[t_0, \infty) \cap C^2(t_0, \infty)$, $f(t_0) > 0$, $f'(t_0) < 0$ and $f''(t) \leq 0$ for $t > t_0$, then there exists a finite positive number $T > t_0$ such that $f(T) = 0$.*

Proof. Seeing that $f \in C^1[t_0, \infty)$ and $f''(t) \leq 0$ for $t > t_0$, we obtain that $f'(t) \leq f'(t_0) < 0$ and $f(t) \leq f(t_0) + f'(t_0)(t - t_0)$. Hence there exists $t_1 > t_0$ such that $f(t_1) < 0$. By the continuity of f in $[t_0, \infty)$, there exists $T \in (t_0, t_1)$ such that $f(T) = 0$. ❑

Lemma 3.4 *Suppose that u is the classical solution of* (2.1). *If $u_0 \geq 0$, $c_2, u_1 > 0$, and $u_0^p \geq -\frac{c_1}{c_2}$, then $u(t), u'(t), u''(t) > 0$ for $t \in [0, T)$, where T is the life-span of u.*

Proof. Suppose that there exists a positive number t_0 such that $u'(t_0) \leq 0$, according to $u \in C^2$ and $u_1 > 0$, then there exists a positive number t_1 defined by

$$t_1 = \inf\{t \in (0, t_0] : u'(t) = 0\}$$

with $u'(t_1) = 0$ and $u'(t) \geq 0$ for $t \in [0, t_1]$. Because $u'(t) \geq 0$ for $t \in [0, t_1]$, we obtain that

$$u(t)^p \geq -\frac{c_1}{c_2} \text{ and } u''(t) \geq 0 \text{ for } t \in [0, t_1].$$

Therefore, $u'(t_1) \geq u_1 > 0$. This result contradicts with $u'(t_1) = 0$; thus we conclude that $u'(t) > 0$ for $t \in [0, T)$, where T is the life-span of u. Together the equation (2.1) and the continuities of u, u' and u'', we obtain the conclusions under this Lemma 3.4. ❑

Together Theorem 2.1 and Theorem 2.2, there exists the unique solution to the (2.1) on $[0, T)$, where T depends on the initial values given by

$$T(u_0, u_1) = \min \left\{ \begin{array}{l} \frac{1}{|u_1|}, \ \frac{1}{|c_1|M^q + |c_2|M^q N^p}, \\ \frac{-|u_1| + \sqrt{u_1^2 + 2(|c_1|M^q + |c_2|M^q N^p)}}{|c_1|M^q + |c_2|M^q N^p}, \\ -1 + \sqrt{1 + \frac{1}{\alpha_1 + \alpha_2 + \alpha_3}} \end{array} \right\}$$

and

$$N = | u_0 | + 1, \ M = | u_1 | + 1,$$

$$\alpha_1 = | c_1 | M^q p N^{p-1}, \ \alpha_2 = | c_1 | q M^{q-1}, \ \alpha_3 = | c_2 | q N^p M^{q-1}.$$

Lemma 3.5 *If* $u_0 \leq u_0^*$ *and* $u_1 \leq u_1^*$, *then* $T(u_0, u_1) \geq T(u_0^*, u_1^*)$.

Proof. Let

$$N^* = | u_0^* | + 1, \ M^* = | u_1^* | + 1,$$

$$\alpha_1^* = | c_1 | M^{*q} p N^{*p-1}, \ \alpha_2^* = | c_1 | q M^{*q-1}, \ \alpha_3^* = | c_2 | q N^{*p} M^{*q-1}.$$

(1) If $T(u_0, u_1) = \dfrac{1}{| u_1 |}$, by $u_1 \leq u_1^*$, then

$$T(u_0, u_1) \geq \frac{1}{| u_1^* |} \geq T(u_0^*, u_1^*).$$

(2) If $T(u_0, u_1) = -1 + \sqrt{1 + \dfrac{1}{\alpha_1 + \alpha_2 + \alpha_3}}$, using the fact that $u_1 \leq u_1^*$ and $p, q \geq 1$ we have $\alpha_1^* \geq \alpha_1 \geq 0$, $\alpha_2^* \geq \alpha_2 \geq 0$ and $\alpha_3^* \geq \alpha_3 \geq 0$. Thus

$$T(u_0, u_1) \geq -1 + \sqrt{1 + \frac{1}{\alpha_1^* + \alpha_2^* + \alpha_3^*}} \geq T(u_0^*, u_1^*).$$

(3) If $T(u_0, u_1) = \dfrac{1}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p}$, according to the conditions $u_0 \le u_0^*$, $u_1 \le u_1^*$ and $p, q \ge 1$ we obtain that $M^{*q} \ge M^q$ and $N^{*p} \ge N^p$ and then

$$T(u_0, u_1) \ge \frac{1}{\mid c_1 \mid M^{*q} + \mid c_2 \mid M^{*q} N^{*p}} \ge T(u_0^*, u_1^*).$$

(4) If $T(u_0, u_1) = \dfrac{-\mid u_1 \mid + \sqrt{u_1^2 + 2(\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p)}}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p}$, from $u_0 \le u_0^*$ and $u_1 \le u_1^*$, it follows that $M^{*q} \ge M^q$, $N^{*p} \ge N^p$ and

$$\begin{aligned} T(u_0, u_1) &= \frac{2}{\mid u_1 \mid + \sqrt{u_1{}^2 + 2(\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p)}} \\ &\ge \frac{2}{\mid u_1^* \mid + \sqrt{u_1^{*2} + 2(\mid c_1 \mid M^{*q} + \mid c_2 \mid M^{*q} N^{*p})}} \\ &\ge T(u_0^*, u_1^*). \ \square \end{aligned}$$

Lemma 3.6 *Suppose that u is the classical solution of (2.1) for $q \in [1, 2]$. If u exists locally and t_1 is the life-span of u, then u blows up at $t = t_1$.*

Proof. Assume that $\lim_{t \to t_1^-} u(t) = M < \infty$. By (3.3), (3.4) and $q \in [1, 2]$, we have

$$\lim_{t \to t_1^-} u'(t) = \begin{cases} [(2-q)(c_1 M + \frac{c_2}{p+1} M^{p+1} + E(0))]^{\frac{1}{2-q}} & \text{if } 1 \le q < 2, \\ \exp\{c_1 M + \frac{c_2}{p+1} M^{p+1} + E_1(0)\} & \text{if } q = 2. \end{cases}$$

Now we consider the following differential equation

$$\begin{cases} v''(t) = v'(t)^q (c_1 + c_2 v(t)^p), \\ v(0) = u(t_1^-), v'(0) = u'(t_1^-). \end{cases}$$

Let $v(t)$ be the existing unique solution to the above equation on $[0, T_v)$. Since $u(t_1^-)$ and $u'(t_1^-)$ are finite, so $T_v > 0$. Let

$$U(t) = \begin{cases} u(t) & \text{if } t \in [0, t_1^-), \\ v(t - t_1^-) & \text{if } t \in [t_1^-, t_1^- + T_v), \end{cases}$$

the problem(2.1) can be solved beyond the time t_1, this contradicts with the assumption of t_1. Therefore, u blows up at $t = t_1$. $\square$

3.1 Blow-up Phenomena of u

To discuss the properties of blow-up phenomena of u with $u_1 \neq 0$, we separate this subsection into three parts $1 \leq q < 2$, $q > 2$ and $q = 2$.

Case1. Blow-up phenomena for $1 \leq q < 2$

In this situation, we have some blow-up results.

Theorem 3.7 *Suppose that u is the classical positive solution of* (2.1) *and $q \in [1, 2)$, $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0,\ u_0^p \geq -\dfrac{c_1}{c_2}$, then u blows up at time $t = T_{11}$ for some finite real number $T_{11} > 0$.*

Remark 3.7 *If we don't restrict ourself to the positiveness of the solution u to the equation* (2.1)*, then we also have the following blow-up results:*
If u is the solution of equation (2.1)*, $q \in [1, 2]$ and one of the followings is valid:*

(1) *p is even, q is odd,* $c_2 > 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \geq -\dfrac{c_1}{c_2}$,

(2) *p is odd, q is even,* $c_2 > 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \leq -\dfrac{c_1}{c_2}$,

(3) *p is even, q is even,* $c_2 < 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \geq -\dfrac{c_1}{c_2}$,

(4) *p is odd, q is odd,* $c_2 < 0,\ u_0 \leq 0,\ u_1 < 0,\ u_0^p \leq -\dfrac{c_1}{c_2}$.

Then u blows up in finite time.

***Proof of Theorem* 3.7.**

Suppose that u is a global solution of equation (2.1).

(I) For $q = 1$, $u''(t) = u'(t)(c_1 + c_2 u(t)^p)$, by (4.1), we have

$$\int_{u_0}^{u(t)} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\, dr = t \quad \text{for all } t > 0.$$

By Lemma 3.4, we have that $u(t) > u_0$ for $t > 0$. Using the fact that $c_1 + \frac{c_2}{p+1} r^{p+1} + E(0) > 0$ for $r \geq u_0$ (see the proof of Theorem 4.2), we get

$$\int_{u_0}^{u(t)} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\, dr \leq \int_{u_0}^{\infty} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\, dr \quad \text{for all } t > 0,$$

and then

$$\int_{u_0}^{\infty} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\, dr \geq \lim_{t\to\infty} \int_{u_0}^{u(t)} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\, dr = \lim_{t\to\infty} t.$$

Since the integral $\displaystyle\int_{u_0}^{\infty} \frac{1}{c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)}\, dr$ is finite (see the proof of Theorem 4.2), thus it results a contradictory conclusion with the above last estimate. So we can conclude that u only exists on $[0, T_{11})$, where T_{11} is the life-span of u. By Lemma 3.6, we obtain that u blows up at $t = T_{11}$.

(II) For $1 < q < 2$, then $m = \frac{1}{2}(\frac{1}{2-q} - 1) > 0$. We claim that there exists a finite time $T_{11} > 0$ such that

$$J(T_{11}) = 0.$$

According to Lemma 4.1, we find that u' and u blow up simultaneously. Thus $u \in C^2[0, T)$, where T is a blow-up time of u. By (3.3) and Lemma 3.4

$$u'(t)^{2-q} = (2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0)) \quad \text{for all } t > 0.$$

By direct computation, we get

$$J'(t) = -ma(t)^{-(m+1)} a'(t) = -ma(t)^{-(m+1)} 2u(t)u'(t),$$

$$\begin{aligned}
& a''(t) \\
=\ & 2u'(t)^2 + 2u(t)u''(t) \\
=\ & 2u'(t)^2 + 2u'(t)^q (c_1 u(t) + c_2 u(t)^{p+1}) \\
=\ & 2u'(t)^2 + 2u'(t)^q (\frac{u'(t)^{2-q}}{2-q} - E(0) + \frac{c_2 p}{p+1} u(t)^{p+1}) \\
=\ & 2(1 + \frac{1}{2-q}) a'(t)^2 + 2u'(t)^q (\frac{c_2 p}{p+1} u(t)^{p+1} - E(0))
\end{aligned}$$

and

$$a(t)a''(t) = \frac{1}{2}(1 + \frac{1}{2-q}) a'(t)^2 + 2a(t)u'(t)^q (\frac{c_2 p}{p+1} u(t)^{p+1} - E(0)).$$

Hence we have

$$\begin{aligned}
& J''(t) \\
=\ & -ma(t)^{-(m+2)} (a(t)a''(t) - (m+1)a'(t)^2) \\
=\ & -ma(t)^{-(m+2)} \{[\frac{1}{2}(1 + \frac{1}{2-q}) - (m+1)] a'(t)^2 + 2a(t)u'(t)^q (\frac{c_2 p}{p+1} u(t)^{p+1} - E(0))\} \\
=\ & -ma(t)^{-(m+2)} 2a(t)u'(t)^q (\frac{c_2 p}{p+1} u(t)^{p+1} - E(0)).
\end{aligned}$$

By Lemma 3.4, we knew that $u(t), u'(t), u''(t) > 0$ for all $t > 0$. Then there exists a finite time $t_1 > 0$ such that

$$\frac{c_2 p}{p+1} u(t_1)^{p+1} - E(0) \geq 0.$$

Herewith, $J(t_1) > 0$, $J'(t_1) < 0$ and $J''(t) \leq 0$ for $t \geq t_1$. Together these and Lemma 3.3 we obtain a finite positive number $T_{11} > t_1$ such that $J(T_{11}) = 0$. Thus u blows up in finite time. Therefore it creates a contradictory result, thus our assumption is a fault. We obtain that u exists locally and by Lemma 3.6, u blows up in finite time.

Proof of Remark 3.7:

Under case (1):

Let $v(t) = -u(t)$. By the fact that p is even and q is odd, we have $v(t)^p = u(t)^p$ and $v'(t)^q = -u'(t)^q$. We get

$$\begin{cases} v''(t) = -u''(t) = -u'(t)^q(c_1 + c_2 u(t)^p) = v'(t)^q(c_1 + c_2 v(t)^p), \\ v(0) = v_0 = -u_0, v'(0) = v_1 = -u_1. \end{cases}$$

Since that $u_0 \leq 0, u_0^p \geq -\dfrac{c_1}{c_2}, u_1 < 0$ and p is even, we have $v_0 \geq 0, v_1 > 0$ and $v_0^p = u_0^p \geq -\dfrac{c_1}{c_2}$. By Theorem 3.7 and Theorem 3.9 below, v blows up, so does u.

To the case (2), we set $v(t) = -u(t)$. Since that p is odd and q is even, $v(t)^p = -u(t)^p$, $v'(t)^q = u'(t)^q$ and

$$\begin{cases} v''(t) = -u''(t) = -u'(t)^q(c_1 + c_2 u(t)^p) = v'(t)^q(-c_1 + c_2 v(t)^p), \\ v(0) = v_0 = -u_0, v'(0) = v_1 = -u_1. \end{cases}$$

According to the condition that $u_0 \leq 0, u_0^p \leq -\dfrac{c_1}{c_2}, u_1 < 0$ and p is odd, we have $v_0 \geq 0, v_1 > 0$ and $v_0^p = -u_0^p \geq \dfrac{c_1}{c_2}$. Using Theorem 3.7 and Theorem 3.9 below, v blows up. Thus u blows up in finite time.

For case (3), let $v(t) = -u(t)$. By the assumption, we have $v(t)^p = u(t)^p$, $v'(t)^q = u'(t)^q$ and

$$\begin{cases} v''(t) = -u''(t) = -u'(t)^q(c_1 + c_2 u(t)^p) = v'(t)^q(-c_1 + (-c_2) v(t)^p), \\ v(0) = v_0 = -u_0, v'(0) = v_1 = -u_1. \end{cases}$$

With the help of the fact that $u_0 \le 0, u_0^p \ge -\dfrac{c_1}{c_2}, u_1 < 0$ and p is even, $v_0 \ge 0, v_1 > 0$ and $v_0^p = u_0^p \ge -\dfrac{c_1}{c_2}$. From Theorem 3.7 and Theorem 3.9 below, it follows that v and u blow up in finite time.

Under the circumstance of (4), let $v(t) = -u(t)$. By the condition that p is odd and q is odd, we have $v(t)^p = -u(t)^p$, $v'(t)^q = -u'(t)^q$ and

$$\begin{cases} v''(t) = -u''(t) = -u'(t)^q(c_1 + c_2 u(t)^p) = v'(t)^q(c_1 + (-c_2)v(t)^p), \\ v(0) = v_0 = -u_0, v'(0) = v_1 = -u_1. \end{cases}$$

Since that $u_0 \le 0, u_0^p \le -\dfrac{c_1}{c_2}, u_1 < 0$ and p is odd, we get that $v_0 \ge 0, v_1 > 0$ and $v_0^p = -u_0^p \ge \dfrac{c_1}{c_2}$. Therefore v and u blow up in finite time. ❑

Now we estimate the blow-up rate and blow-up constant, we have:

Theorem 3.8 *Suppose that u is a classical solution of* (2.1). *If* $1 \le q < 2$ *and u blows up in finite time, then the blow-up rate of u is* $\dfrac{2-q}{p+q-1}$ *and the blow-up constant of u is* $(\dfrac{p+q-1}{2-q})^{-\frac{2-q}{p+q-1}}[(2-q)\dfrac{c_2}{p+1}]^{\frac{-1}{p+q-1}}$.

Proof. Let $i = \dfrac{p+q-1}{2-q}$, by some calculations and (2.1) using L.Hôpital's rule we obtain

$$\begin{aligned} &\lim_{t \to T_{11}^-} \frac{u^{-i}}{T_{11} - t} \\ = &\lim_{t \to T_{11}^-} i u(t)^{-(i+1)} u'(t) \\ = &\lim_{t \to T_{11}^-} i \frac{[(2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0))]^{\frac{1}{2-q}}}{u(t)^{i+1}} \\ = &\frac{p+q-1}{2-q}[(2-q)\frac{c_2}{p+1}]^{\frac{1}{2-q}}. \end{aligned}$$

Thus

$$\lim_{t \to T_{11}^-} (T_{11} - t)^{\frac{2-q}{p+q-1}} u(t) = (\frac{p+q-1}{2-q})^{-\frac{2-q}{p+q-1}}[(2-q)\frac{c_2}{p+1}]^{\frac{-1}{p+q-1}}. ❑$$

Case2. Blow-up Phenomena for $q = 2$

In the particular case, $q = 2$, we obtain an interesting blow-up result and especial blow-up constant.

Theorem 3.9 *For $q=2$, if u is the classical positive solution of* (2.1) *and* $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0,\ u_0^p \geq -\frac{c_1}{c_2}$. *Then u blows up logarithmically at finite time $t = T_{12}$ and*

$$\lim_{t\to T_{12}^-}[\frac{1}{-\ln(T_{12}-t)}]^{\frac{1}{p+1}}u(t) = [\frac{c_2}{p+1}]^{-\frac{1}{p+1}}.$$

Proof. Assume that u is a global solution of (2.1). By (3.4) and Lemma 3.4,

$$\ln|\,u'(t)\,| = (c_1u(t) + \frac{c_2}{p+1}u(t)^{p+1} + E_1(0)) \quad \text{for all } t > 0.$$

Since that $u(t)$, $u'(t)$ blow up contemporarily (see Lemma 4.1), $u \in C^2[0, T_{12})$ where T_{12} is blow-up time of u.

Let $K(t) = a(t)^{-1}$, then

$$K'(t) = -a(t)^{-2}a'(t) = -2a(t)^{-2}u(t)u'(t)$$

and

$$\begin{aligned} K''(t) &= -a(t)^{-3}(a(t)a''(t) - 2a'(t)^2) \\ &= -a(t)^{-3}[2a(t)(u'(t)^2 + u(t)u''(t)) - 2a'(t)^2] \\ &= -a(t)^{-3}\{2a(t)[u'(t)^2 + u(t)u'(t)^2(c_1 + c_2u(t)^p)] - 2a'(t)^2\} \\ &= -a(t)^{-3}\{2a(t)u'(t)^2[1 + u(t)(c_1 + c_2u(t)^p)] - 2a'(t)^2\} \\ &= -a(t)^{-3}\{\frac{1}{2}a'(t)^2[1 + u(t)(c_1 + c_2u(t)^p)] - 2a'(t)^2\} \\ &= -a(t)^{-3}a'(t)^2\{\frac{1}{2}[1 + u(t)(c_1 + c_2u(t)^p)] - 2\}. \end{aligned}$$

By Lemma 3.4, $u(t), u'(t), u''(t) > 0$ for $t > 0$. Hence there exists $t_0 > 0$ such that

$$u(t) \geq (\frac{|\,c_1\,| + 3}{c_2})^{\frac{1}{p}} + 1 \quad \text{for } t \geq t_0.$$

Thus we have

$$\frac{1}{2}[(1 + u(t)(c_1 + c_2u(t)^p)] - 2 \geq 0 \quad \text{for } t \geq t_0.$$

We conclude that

$$K(t_0) > 0, K'(t) < 0 \text{ and } K''(t) < 0 \quad \text{for } t \geq t_0,$$

thus by Theorem 3.3 there exists positive number T_{12} such that $K(T_{12}) = 0$ and u blows up at time $t = T_{12}$. This result contradicts with our assumption that u is a global solution of problem (2.1). Therefore u exists locally. By Lemma 3.6, u blows up in finite time. After some computations we get

$$\begin{aligned}\lim_{t\to T_{12}^-} -\ln(T_{12}-t)u(t)^{-(p+1)} &= \lim_{t\to T_{12}^-} \frac{-\ln(T_{12}-t)}{u(t)^{p+1}} \\ &= \lim_{t\to T_{12}^-} \frac{\frac{1}{T_{12}-t}}{(p+1)u(t)^p u'(t)} \\ &= \lim_{t\to T_{12}^-} \frac{u(t)^{-p}u'(t)^{-1}}{(p+1)(T_{12}-t)} \\ &= \lim_{t\to T_{12}^-} \frac{pu(t)^{-(p+1)} + u(t)^{-p}u'(t)^{-2}u''(t)}{p+1}.\end{aligned}$$

Using (2.1), we obtain $u''(t) = u'(t)^2(c_1 + c_2 u(t)^p)$ and

$$\begin{aligned}\lim_{t\to T_{12}^-} -\ln(T_{12}-t)u(t)^{-(p+1)} &= \lim_{t\to T_{12}^-} \frac{pu(t)^{-(p+1)} + u(t)^{-p}(c_1 + c_2 u(t)^p)}{p+1} \\ &= \frac{c_2}{p+1}.\end{aligned}$$

Hence we conclude

$$\lim_{t\to T_{12}^-} [\frac{1}{-\ln(T_{12}-t)}]^{\frac{1}{p+1}} u(t) = [\frac{c_2}{p+1}]^{-\frac{1}{p+1}}. \ ❑$$

Case3. Blow-up phenomena for $q > 2$

Under $q > 2$ we have the boundedness for the solution.

Theorem 3.10 *For $q > 2$, if u is the classical positive solution of (2.1) and $c_2 > 0$, $u_0 \geq 0$, $u_1 > 0$, $u_0^p \geq -\frac{c_1}{c_2}$, then u is bounded in $[0, T)$, where T is the life span of u.*

Proof. We integrate the equation (2.1) from 0 to t and then we obtain

$$\frac{u'(t)^{2-q}}{2-q} - \frac{u_1^{2-q}}{2-q} = c_1 u(t) + \frac{c_2}{p+1}u(t)^{p+1} - c_1 u_0 - \frac{c_2}{p+1}u_0^{p+1}.$$

For $t \in [0, T)$, by Lemma 3.4, then $u(t), u'(t) > 0$ and

$$\frac{u_1^{2-q}}{q-2} > c_1 u(t) + \frac{c_2}{p+1}u(t)^{p+1} - c_1 u_0 - \frac{c_2}{p+1}u_0^{p+1}.$$

Since that $c_2 > 0$ and $u(t) > 0$ for $t \in [0, T)$, u is bounded in $[0, T)$. ❑

3.2 Blow-up Phenomena of u'

In this subsection we come back to the consideration of blow-up phenomena of u', and we have

Theorem 3.11 *For $q \geq 1$, if u is a classical positive solution of* (2.1) *and* $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0,\ u_0^p \geq -\frac{c_1}{c_2}$, *then u' blows up at time $t = T_2$.*

Proof. We separate this proof into three parts $1 \leq q < 2$, $q = 2$ and $q > 2$.

(I) At first, we assume that $1 \leq q < 2$, by Theorem 3.7 and Lemma 4.1 below, then u and u' blow up in finite time.

(II) For $q = 2$, using Theorem 3.9 and Lemma 4.1 below, then u and u' blow up in finite time.

(III) Assume that $q > 2$, let

$$b(t) = u'(t)^2,\ L(t) = b(t)^{-\alpha},$$

where $\alpha = \frac{1}{2}(q-1)$, we have

$$L'(t) = -\alpha b(t)^{-(\alpha+1)} b'(t) = -2\alpha b(t)^{-(\alpha+1)} u'(t) u''(t),$$

and

$$\begin{aligned}
L''(t) &= -\alpha b(t)^{-(\alpha+2)}[b(t)b''(t) - (\alpha+1)b'(t)^2] \\
&= -\alpha b(t)^{-(\alpha+2)}[b(t)(2u''(t)^2 + 2u'(t)u'''(t)) - (\alpha+1)b'(t)^2] \\
&= -\alpha b(t)^{-(\alpha+2)}[b(t)(2u''(t)^2 + 2qu''(t)^2 + 2pc_2u(t)^{p-1}u'(t)^{q+2}) - (\alpha+1)b'(t)^2] \\
&= -\alpha b(t)^{-(\alpha+2)}[(\frac{1}{2}(1+q) - (\alpha+1))b'(t)^2 + 2c_2pb(t)u(t)^{p-1}u'(t)^{q+2}] \\
&= -2pc_2\alpha b(t)^{-(\alpha+1)}u(t)^{p-1}u'(t)^{q+2}.
\end{aligned}$$

By Lemma 3.4, $u(t) > 0$, $u'(t) > 0$ and $u''(t) > 0$ for $t > 0$, and then we obtain that $L'(t), L''(t) < 0$ for $t > 0$. Now we need to check that u doesn't blow up earlier than

u'. By Theorem 3.10, u is bounded. Using Lemma 3.3, there exists a finite number T_2 such that $L(T_2) = 0$. Since that $q > 2$, thus $\alpha > 0$. We obtain that u' blows up at finite time $t = T_2$. ❑

Having obtained the blow-up phenomena of u', we want to calculate blow-up rate and blow-up constant of u'.

Case1. Blow-up rates and blow-up constants of u' for $1 \le q < 2$

For $q \in [1, 2)$ we have the result:

Theorem 3.12 *Under the conditions in Theorem 3.11, for $1 \le q < 2$, u' blows up in finite time with blow-up rate $\dfrac{p+1}{p+q-1}$ and blow-up constant*

$$[\frac{c_2(p+q-1)}{p+1}(\frac{c_2(2-q)}{p+1})^{\frac{-p}{p+1}}]^{\frac{-(p+1)}{p+q-1}}.$$

Proof. By Lemma 4.1 u and u' have the same blow-up time. According to (2.1), L.Hôpital's rule and Theorem 3.8 we have

$$\begin{aligned}\lim_{t\to T_2^-}\frac{u'(t)^{\frac{1-p-q}{p+1}}}{(T_2-t)} &= \lim_{t\to T_2^-}\frac{p+q-1}{p+1}u'(t)^{\frac{-(2p+q)}{p+1}}u''(t)\\ &= \lim_{t\to T_2^-}\frac{c_2(p+q-1)}{p+1}[(2-q)(c_1u(t)+\frac{c_2}{p+1}u(t)^{p+1}+E(0))]^{\frac{-p}{p+1}}u(t)^p\\ &= \frac{c_2(p+q-1)}{p+1}(\frac{c_2(2-q)}{p+1})^{\frac{-p}{p+1}}.\end{aligned}$$

Thus

$$\lim_{t\to T_2^-}(T_2-t)^{\frac{p+1}{p+q-1}}u'(t) = [\frac{c_2(p+q-1)}{p+1}(\frac{c_2(2-q)}{p+1})^{\frac{-p}{p+1}}]^{\frac{-(p+1)}{p+q-1}}. \ ❑$$

Case2. Blow-up rates and blow-up constants of u' for $q = 2$

To the case $q = 2$, we have the following results on blow-up rate and blow-up constant for u'.

Theorem 3.13 *Under the conditions in Theorem 3.11, for $q = 2$, u' blows up in finite time, we also have*

$$\lim_{t\to T_2^-}[-\ln(T_2-t)]^{\frac{p}{p+1}}(T_2-t)u'(t) = c_2^{\frac{-1}{p+1}}(\frac{1}{p+1})^{\frac{p}{p+1}}.$$

Proof. According to Lemma 4.1 u and u' have the same life-span. By (2.1), L.Hôpital's rule and Theorem 3.9 we have

$$\begin{aligned}
& \lim_{t\to T_2^-} [-\ln(T_2-t)]^{\frac{p}{p+1}}(T_2-t)u'(t) \\
= & \lim_{t\to T_2^-} \frac{[-\ln(T_2-t)]^{\frac{p}{p+1}}(T_2-t)}{u'(t)^{-1}} \\
= & \lim_{t\to T_2^-} \frac{\frac{p}{p+1}[-\ln(T_2-t)]^{\frac{-1}{p+1}}(T_2-t)-[-\ln(T_2-t)]^{\frac{p}{p+1}}}{-(c_1+c_2u(t)^p)} \\
= & c_2^{\frac{-1}{p+1}}(\frac{1}{p+1})^{\frac{p}{p+1}}. \;❑
\end{aligned}$$

Case3. Blow-up rates and blow-up constants of u' for $q>2$

In this case of $q>2$ we also have the blow-up result for u'.

Theorem 3.14 *Under the conditions in Theorem* 3.11, *for* $q>2$, u' *blows up in finite time with blow-up rate* $\dfrac{1}{q-1}$ *and blow-up constant* $[(q-1)(c_1+c_2u(T_2)^p)]^{\frac{1}{1-q}}$.

Proof. For $q>2$, by (2.1) and L.Hôpital's rule we have

$$\begin{aligned}
\lim_{t\to T_2^-}\frac{u'(t)^{1-q}}{(T_2-t)} &= \lim_{t\to T_2^-}(1-q)u'(t)^{-q}u''(t)(-1) \\
&= \lim_{t\to T_2^-}(q-1)(c_1+c_2u(t)^p) \\
&= (q-1)(c_1+c_2u(T_2)^p).
\end{aligned}$$

Thus

$$\lim_{t\to T_2^-}(T_2-t)^{\frac{1}{q-1}}u'(t)=[(q-1)(c_1+c_2u(T_2)^p)]^{\frac{1}{1-q}}. \;❑$$

In the coming subsection we treat the blow-up phenomena of u'' under three cases $1\le q<2$, $q=2$ and $q>2$.

3.3 Blow-up Phenomena of u''

We want to calculate blow-up rate and blow-up constant of u'' in the this subsection.

Theorem 3.15 *Suppose that u is a classical positive solution of* (2.1). *If $1 \le q$, then u'' blows up at time $t = T_3$ under the same conditions in Theorem* 3.11.

Proof. According to Theorem 3.11 and Lemma 4.1 below, u' and u'' blow up at the same time, $t = T_3$. ❑

Case1. Blow-up rates and blow-up constants of u'' for $1 \le q < 2$

Theorem 3.16 *Under the conditions in Theorem* 3.15, *for $1 \le q < 2$, u'' blows up in finite time with the blow-up rate $\frac{q(p+1)}{p+q-1} + \frac{p(2-q)}{p+q-1}$ and the blow-up constant*

$$c_2\{[\frac{c_2(p+q-1)}{p+1}(\frac{c_2(2-q)}{p+1})^{\frac{-p}{p+1}}]^{\frac{-(p+1)}{p+q-1}}\}^q\{(\frac{p+q-1}{2-q})^{-\frac{2-q}{p+q-1}}[(2-q)\frac{c_2}{p+1}]^{\frac{-1}{p+q-1}}\}^p.$$

Proof. For $1 \le q < 2$, by Lemma 4.1, u, u' and u'' possess the same blow-up time . Using (2.1) again, Theorem 3.8 and Theorem 3.12, we conclude that

$$\begin{aligned} & \lim_{t \to T_3^-} (T_3 - t)^{\frac{q(p+1)}{p+q-1} + \frac{p(2-q)}{p+q-1}} u''(t) \\ = & \lim_{t \to T_3^-} (T_3 - t)^{\frac{q(p+1)}{p+q-1}} u'(t)^q (T_3 - t)^{\frac{p(2-q)}{p+q-1}} (c_1 + c_2 u(t)^p) \\ = & \; c_2\{[\frac{c_2(p+q-1)}{p+1}(\frac{c_2(2-q)}{p+1})^{\frac{-p}{p+1}}]^{\frac{-(p+1)}{p+q-1}}\}^q\{(\frac{p+q-1}{2-q})^{-\frac{2-q}{p+q-1}}[(2-q)\frac{c_2}{p+1}]^{\frac{-1}{p+q-1}}\}^p. \end{aligned}$$ ❑

Case2. Blow-up rates and blow-up constants of u'' for $q = 2$

Theorem 3.17 *Under the conditions in Theorem* 3.15, *for $q = 2$, u'' blows up in finite time and*

$$\begin{aligned} & \lim_{t \to T_3^-} \{[-\ln(T_3 - t)]^{\frac{p}{p+1}}(T_3 - t)\}^q \{[-\ln(T_3 - t)]^{\frac{-1}{p+1}}\}^p u''(t) \\ = & \; c_2[c_2^{\frac{-1}{p+1}}(\frac{1}{p+1})^{\frac{p}{p+1}}]^q[(\frac{c_2}{p+1})^{\frac{-1}{p+1}}]^p. \end{aligned}$$

Proof. For $q = 2$, using Lemma 4.1, u, u' and u'' have the same blow-up time. Thus T_3 is also blow-up time of u and u'. By (2.1), Theorem 3.9 and Theorem 3.13

we conclude that

$$
\begin{aligned}
&\lim_{t\to T_3^-}\{[-\ln(T_3-t)]^{\frac{p}{p+1}}(T_3-t)\}^q\{[-\ln(T_3-t)]^{\frac{-1}{p+1}}\}^p u''(t)\\
=\ &\lim_{t\to T_3^-}\{[-\ln(T_3-t)]^{\frac{p}{p+1}}(T_3-t)\}^q u'(t)^q\{[-\ln(T_3-t)]^{\frac{-1}{p+1}}\}^p(c_1+c_2u(t)^p)\\
=\ &c_2[c_2^{\frac{-1}{p+1}}(\frac{1}{p+1})^{\frac{p}{p+1}}]^q[(\frac{c_2}{p+1})^{\frac{-1}{p+1}}]^p. \ \square
\end{aligned}
$$

Case3. Blow-up rates and blow-up constants of u'' for $q>2$

Theorem 3.18 *Under the conditions in Theorem 3.15, for $q>2$, u'' blows up time in finite time with the blow-up rate $\dfrac{q}{q-1}$ and the blow-up constant*

$$(c_1+c_2u(T_3)^p)\{[(q-1)(c_1+c_2u(T_3)^p)]^{\frac{1}{1-q}}\}^q.$$

Proof. For $q>2$, by Lemma 4.1, u'' and u' blow up contemporaneously in finite time. Thanks to Lemma 3.4 we have $u(t)>0$ and $u(t)^p\geq -\frac{c_1}{c_2}$. Since $c_2>0$, $c_1+c_2u(t)^p>0$. By (2.1) and Theorem 3.14, we conclude that

$$
\begin{aligned}
&\lim_{t\to T_3^-}(T_3-t)^{\frac{q}{q-1}}u''(t)\\
=\ &\lim_{t\to T_3^-}(T_3-t)^{\frac{q}{q-1}}u'(t)^q(c_1+c_2u(t)^p)\\
=\ &(c_1+c_2u(T_3)^p)\{[(q-1)(c_1+c_2u(T_3)^p)]^{\frac{1}{1-q}}\}^q. \ \square
\end{aligned}
$$