4 Estimations for the Life-Spans

To estimate the life-span of the solution of the equation (2.1), we separate this section into two parts, $1 \leq q < 2$ and $q = 2$. Here the life-span T of u means that u is the solution of problem (2.1) and the existence interval of u is contained only in $[0, T)$ so that the problem (2.1) has the solution $u \in C^2[0, T)$. We have the following results.

Lemma 4.1 *Suppose that $u \in C^2[0,T)$ is a classical positive solution of problem (2.1) and that $c_2 > 0,\ u_0 \geq 0,\ u_1 > 0,\ u_0^p \geq \frac{-c_1}{c_2}$. For $1 \leq q \leq 2$, $u(t)$ and $u'(t)$ blow up simultaneously; further so does u''. For $q > 2$, $u'(t)$ and u'' blow up at the same time.*

Proof.

(I) For $1 \leq q < 2$, by (3.3) we have

$$u'(t)^{2-q} = (2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0)).$$

(1) At first, we claim that if u blows up in finite time, so does u'. According to Theorem 3.7, u blows up at time $t = T_{11}$. Since $\lim_{t \to T_{11}^-} \frac{1}{u(t)} = 0$, we have

$$\begin{aligned} \lim_{t \to T_{11}^-} \frac{1}{u'(t)^{2-q}} &= \lim_{t \to T_{11}^-} \frac{1}{(2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0))} \\ &= \lim_{t \to T_{11}^-} \frac{\frac{1}{u(t)^{p+1}}}{(2-q)(\frac{c_1}{u(t)^p} + \frac{c_2}{p+1} + \frac{E(0)}{u(t)^{p+1}})} \\ &= 0. \end{aligned}$$

Therefore,

$$\lim_{t \to T_{11}^-} \frac{1}{u'(t)} = 0.$$

Thus, u' blows up in finite time.

(2) We claim that if u' blows up in finite time, then so does u. With the help

of Theorem 3.11, u' blows up at time $t = T_2$. Assume that u doesn't blow up at time $t = T_2$. Let

$$\lim_{t \to T_2^-} u(t) = M < \infty.$$

Then

$$\begin{aligned}
\lim_{t \to T_2^-} u'(t)^{2-q} &= \lim_{t \to T_2^-} (2-q)(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E(0)) \\
&= (2-q)(c_1 M + \frac{c_2}{p+1} M^{p+1} + E(0)) \\
&< \infty.
\end{aligned}$$

This result contradicts with the fact that $u'(t)$ blows up at time $t = T_2$. It deduces that u blows up at time $t = T_2$. Associate (1) with (2), we conclude that u and u' blow up simultaneously.

(II) For the case $q = 2$, by (3.4) we have

$$\ln | u'(t) | = c_1 u(t) + \frac{c_2}{p+1} u(t)^p + E_1(0).$$

(3) We claim that if u blows up in finite time, then so does u'.
By Theorem 3.9 and Lemma 3.4, u blows up at time $t = T_{12}$ and $u(t), u'(t) > 0$. Since that $c_2 > 0$ and u blows up toward positive direction, $\ln | u' |$ also blows up toward positive direction. Thus u' blows up at time $t = T_{12}$.
(4) To prove that u' blows up then so does u. Using Theorem 3.11 and Lemma 3.4, u' blows up at time $t = T_2$ and $u(t), u'(t) > 0$. Assume that u doesn't blow up at time $t = T_2$. Let

$$\lim_{t \to T_2^-} u(t) = M < \infty.$$

Then

$$\begin{aligned}
\lim_{t \to T_2^-} \ln | u'(t) | &= \lim_{t \to T_2^-} (c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E_1(0)) \\
&= (2-q)(c_1 M + \frac{c_2}{p+1} M^{p+1} + E_1(0)) \\
&< \infty.
\end{aligned}$$

This result is contradictory to the fact that u' blows up in finite time. It deduces that u blows up at time $t = T_2$. Together (3) and (4), we conclude that u and u' blow up simultaneously. From (2.1), we have

$$u''(t) = u'(t)^q(c_1 + c_2u(t)^P).$$

Since that u and u' blow up toward positive direction at the same time and $c_2 > 0$. Thus u'' blows up toward positive direction.

(III) Under $q > 2$, according to Theorem 3.11, u' blows up at time $t = T_2$. By Theorem 3.10, we obtain that u is bounded in $[0, T_2)$ and by Lemma 3.4 we have $u'(t) > 0$ for $t \in [0, T_2)$. So the following limit exists,

$$\lim_{t \to T_2^-} c_1 + c_2u(t)^p.$$

Since that $u_0 \geq \dfrac{-c_1}{c_2}$ and $u'(t) > 0$ for $t \in [0, T_2)$,

$$\lim_{t \to T_2^-} c_1 + c_2u(t)^p > 0.$$

By $u''(t) = u'(t)^q(c_1 + c_2u(t)^p)$, it deduces that u' and u'' blow up simultaneously. ❑

Case1. Life-Span for $1 \leq q < 2$

Theorem 4.2 *Suppose that $u \in C^2[0, T)$ is the classical positive solution of (2.1) and T is life-span of u and that T_1 is blow-up time of u. Under the same conditions in Theorem 3.7, T is bounded. We have the estimation*

$$T \leq T_1 = (2-q)^{\frac{1}{q-2}} \int_{u_0}^{\infty} (c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{q-2}}\, dr.$$

Proof. Since that $1 \leq q < 2$, by (3.3) we know that

$$u'(t)^{2-q} = (2-q)(c_1u(t) + \frac{c_2}{p+1}u(t)^{p+1} + E(0)).$$

Using the fact that $u'(t) > 0$ for $t \in [0, T_1)$ and $u'(t) = [(2-q)(c_1u(t) + \frac{c_2}{p+1}u(t)^{p+1} + E(0))]^{\frac{1}{2-q}}$, we have

$$\frac{u'(t)}{(c_1u(t) + \frac{c_2}{p+1}u(t)^{p+1} + E(0))^{\frac{1}{2-q}}} = (2-q)^{\frac{1}{2-q}}.$$

Integrate the last equation from 0 to t, we obtain that

$$\int_0^t \frac{u'(r)}{(c_1 u + \frac{c_2}{p+1} u^{p+1} + E(0))^{\frac{1}{2-q}}(r)}\, dr = (2-q)^{\frac{1}{2-q}} t,$$

$$\int_{u_0}^{u(t)} \frac{1}{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{2-q}}}\, dr = (2-q)^{\frac{1}{2-q}} t. \tag{4.1}$$

Let

$$T_1 = (2-q)^{\frac{1}{q-2}} \int_{u_0}^{\infty} (c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{q-2}}\, dr.$$

We claim that $T_1 < \infty$. By $u_0 \geq (\frac{-c_1}{c_2})^{\frac{1}{p}}$ and

$$c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0) = \int_{u_0}^{r} (c_1 + c_2 s^p)\, ds + \frac{u_1^{2-q}}{2-q},$$

we obtain

$$c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0) > 0 \text{ for } r \geq u_0.$$

And $c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0)$ is continuous on $[u_0, a]$ for $a \geq u_0$. So the function $\frac{1}{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{2-q}}}$ is integrable and positive on $[u_0, a]$ for $a \geq u_0$. We calculate the limit

$$\lim_{r\to\infty} \frac{\frac{1}{r^{\frac{p+1}{2-q}}}}{\frac{1}{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{2-q}}}} = \lim_{r\to\infty} (c_1 r^{-p} + \frac{c_2}{p+1} + E(0) r^{-(p+1)})^{\frac{1}{2-q}}$$

$$= (\frac{c_2}{p+1})^{\frac{1}{2-q}} > 0.$$

By $\frac{p+1}{2-q} > 2$, we gain $\int_{u_0}^{\infty} \frac{1}{r^{\frac{p+1}{2-q}}}\, dr < \infty$ and

$$\int_{u_0}^{\infty} (c_1 r + \frac{c_2}{p+1} r^{p+1} + E(0))^{\frac{1}{q-2}}\, dr < \infty.$$

Thus $T_1 < \infty$. Since that $u \in C^2[0, T)$, $T \leq T_1$. ❑

Case2. Life-Span for $q = 2$

Theorem 4.3 *For $q = 2$, if $u \in C^2[0, T)$ is the classical positive solution of (2.1) and if $c_2 > 0$, $u_0, u_1 > 0$, $u_0^p \geq -\frac{c_1}{c_2}$. Suppose that T is life-span and T_1^* is blow-up time of u. Then T is bounded. We have the estimation*

$$T \leq T_1^* = \int_{u_0}^{\infty} \frac{1}{\exp{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0))}}\, dr.$$

Proof. For $q = 2$ by (3.4),

$$\ln | u'(t) | = c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E_1(0).$$

By $u'(t) > 0$ and $u'(t) = \exp\left(c_1 u(t) + \dfrac{c_2}{p+1} u(t)^{p+1} + E_1(0)\right)$, we have

$$\frac{u'(t)}{\exp\left(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E_1(0)\right)} = 1.$$

Integrate the above equation from 0 to t, we obtain

$$\int_0^t \frac{u'(t)}{\exp\left(c_1 u(t) + \frac{c_2}{p+1} u(t)^{p+1} + E_1(0)\right)}\, dr = t$$

and then

$$\int_{u_0}^{u(t)} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\, dr = t.$$

Let

$$T_1^* = \int_{u_0}^{\infty} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\, dr.$$

We claim that $T_1^* < \infty$. Let

$$f(r) = c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0).$$

Then $f'(r) \geq 0$ for $r^p \geq \dfrac{-c_1}{c_2}$ and $f''(r) \geq 0$ for $r \geq 0$. So there exists $r_0 > 0$ and $r_0^p \geq \dfrac{-c_1}{c_2}$ such that $f(r) > 0$ for $r \geq r_0$.

We calculate

$$\begin{aligned} & \int_{u_0}^{\infty} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\, dr \\ = & \int_{u_0}^{r_0} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\, dr + \int_{r_0}^{\infty} \frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}\, dr \end{aligned}$$

Since that $\dfrac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)}$ is a continuous function on $[u_0, r_0]$, the first integrand is bounded. Since that

$$\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right) > c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0) > 0 \text{ for } r \geq r_0,$$

we obtain

$$\frac{1}{\exp\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right)} < \frac{1}{\left(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)\right.} \text{ for } r \geq r_0.$$

By $\int_{r_0}^{\infty} \frac{1}{(c_1 r + \frac{c_2}{p+1} r^{p+1} + E_1(0)} \, dr < \infty$, and the comparison test, the second integrand is bounded. Therefore, T_1^* is bounded . Since that $u \in C^2(0,T)$, therefore $T \leq T_1^*$. ❑