1 Introduction

In the last thirty years, the problem of solving $\bar{\partial}$-equation, namely $\bar{\partial}u = f$, is an important research area in the theory of several complex variables. Of course, a bunch of famous results appear during the times. Now we just list some of them which we are interested in.

Given a domain Ω in $\mathbb{C}^n$, $n \geq 2$. The $\bar{\partial}$-equation can be formulated as follows: Let $f = \sum_{j=1}^{n} f_j d\bar{z}_j$ be a $\bar{\partial}$-closed $(0,1)$-form on Ω, i.e., $\bar{\partial}f = 0$ on Ω. Can one find a function u on Ω such that $\bar{\partial}u = f$ on Ω ?

In 1965, Hörmander [7] proved that the $\bar{\partial}$-equation is solvable for pseudoconvex domain Ω. Moreover, if $\Omega \subseteq \mathbb{C}^n$ is a bounded pseudoconvex domain and $f \in L^2_{(0,1)}(\Omega)$, the space of $(0,1)$-forms with coefficients in $L^2(\Omega)$, is a $\bar{\partial}$-closed $(0,1)$-form on Ω, then there exists $u \in L^2(\Omega)$ such that

$$\bar{\partial}u = f \quad \text{on } \Omega$$

and

$$\|u\|_2 \leq C_\Omega \|f\|_2,$$

where $C_\Omega > 0$ is a constant depending only on Ω. Hörmander's results can be used to solve the famous Levi problem [7,8] in complex function theory, namely, every pseudoconvex domain is a domain of holomorphy. On the other hand, it motivates the following problem:

Given $1 \leq p \leq \infty$ and $f \in L^p_{(0,1)}(\Omega)$ with $\bar{\partial}f = 0$, where $L^p_{(0,1)}(\Omega)$ denotes the space of $(0,1)$-forms with coefficients in $L^p(\Omega)$. Can one solve $\bar{\partial}u = f$ with L^p-estimate, i.e., $\|u\|_p \leq C_\Omega \|f\|_p$ for some constant $C_\Omega > 0$?

The complete answer was proved by Kerzman [9] for strongly pseudoconvex domains with smooth boundary by using local construction. In 1970, Henkin [5], Ramirez

[12] and Grauert & Lieb [4] independently constructed solution of $\bar{\partial} u = f$ on strongly pseudoconvex domain with smooth boundary by using integral representation. Among these constructions, Henkin's result is the most important one because he obtained an explicit solution of $\bar{\partial}$-equation and provided an uniform estimate of the solution, namely $\|u\|_\infty \leq C_\Omega \|f\|_\infty$, where C_Ω is a constant depending only on the domain Ω.

There are two major steps in the construction of Henkin's solution to $\bar{\partial} u = f$. One is how to construct the Henkin's kernel in the integral representation of u and the other is to see the relation between the constant C_Ω and the domain Ω.

Since the Henkin's kernel involves a series of non-constructive steps, there is no way to get an explicit upper bound for C_Ω for arbitrary strongly pseudoconvex domain. However, for some special class of domains, for examples, unit balls, ellipsoids and strictly convex domains, the Henkin's kernel and C_Ω can be written down explicitly, see [1,2,3].

In this thesis, we are interested in the Henkin's solution of $\bar{\partial} u = f$ and the uniform estimate $\|u\|_\infty \leq C_S \|f\|_\infty$ on arbitrary shell domain $S = \{\, z \in \mathbb{C}^n \mid R_1^2 < \sum_{j=1}^{n} |\, z_j \,|^2 < R_2^2 \,\}$, where $0 < R_1 < R_2$. Note that the Henkin's kernel was mentioned in [13] in this case.

We will follow the main idea and techniques used in [1] to obtain the solution of $\bar{\partial} u = f$ and an explicit upper bound for the constant C_S.

The thesis contains five sections. In section 1, we give an introduction. In section 2, we review some general results. In section 3, we write down the integral representation of the solution of $\bar{\partial} u = f$ for arbitrary ball on $\mathbb{C}^n$. In section 4, we give an uniform estimate for the solution $\bar{\partial} u = f$ constructed in section 3. In section 5, we prove our main result, i.e., for arbitrary shell domain, we can write down the explicit solution of $\bar{\partial} u = f$ with uniform estimate $\|u\|_\infty \leq C_S \|f\|_\infty$.

2 General Results

Let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ be the n-dimensional complex Euclidean space. $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$ in a natural way; that is,

$$(z_1, z_2, \ldots, z_n) = (x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n) \approx (x_1, y_1, x_2, y_2, \ldots, x_n, y_n).$$

The standard Euclidean volume form on $\mathbb{C}^n$ is given by

$$\begin{aligned} dV(z) &= (\frac{1}{2i})^n (d\bar{z}_1 \wedge dz_1) \wedge \cdots \wedge (d\bar{z}_n \wedge dz_n) \\ &= (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n). \end{aligned}$$

Likewise, the symbol $d\sigma$ denotes the Euclidean $(2n-1)$-dimensional surface measure on the boundary of the unit ball B, i.e., the volume form of $\partial B = S^{2n-1}$.

The operator $\bar{\partial}$ is the differential operator which associates a form of degree $(0, q)$ to the form of degree $(0, q+1)$; more precisely, if $f(z) = \sum_{|I|=q} f_I(z) d\bar{z}^I$, then

$$\bar{\partial} f = \sum_I \sum_{j=1} \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}^I.$$

Also, we introduce differential operators $\bar{\partial}_\zeta$, $\bar{\partial}_{z,\zeta}$ and $d_{\zeta,\lambda}$ as follows:

$$\bar{\partial}_\zeta g(z, \zeta) = \sum_{j=1}^n \frac{\partial g(z, \zeta)}{\partial \bar{\zeta}_j} d\bar{\zeta}_j,$$

$$\bar{\partial}_{z,\zeta} g(z, \zeta) = \sum_{j=1}^n \frac{\partial g(z, \zeta)}{\partial \bar{z}_j} d\bar{z}_j + \sum_{j=1}^n \frac{\partial g(z, \zeta)}{\partial \bar{\zeta}_j} d\bar{\zeta}_j$$

and

$$d_{\zeta,\lambda} = \partial_{\zeta,\lambda} + \bar{\partial}_{\zeta,\lambda}.$$

Besides, we let the symbol $\langle \ , \ \rangle$ denote the standard inner product on $\mathbb{C}^n$. In addition, we put $\langle \ \cdot \ \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by $\langle z \cdot w \rangle = \sum_{j=1}^n z_j w_j$, where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Definition 2.1 *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $k = 1, 2, \ldots, \infty$ or ω. Ω is said to have C^k boundary if there is a C^k function $\rho : \mathbb{C}^n \to \mathbb{R}$ satisfying the following properties:*

1. $\Omega = \{ z \in \mathbb{C}^n \mid \rho(z) < 0 \}$.

2. $\bar{\Omega}^c = \{ z \in \mathbb{C}^n \mid \rho(z) > 0 \}$.

3. $\operatorname{grad} \rho(z) = \nabla \rho(z) \neq 0$ *for all* $z \in \partial\Omega$.

Such ρ is called a C^k defining function for Ω.

Definition 2.2 *Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with C^k boundary, where $k \geq 2$. Let ρ be a defining function of Ω. A point $p \in \partial\Omega$ is called pseudoconvex if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0 \tag{2.1}$$

for all $w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$ with

$$\langle \nabla \rho(p) \cdot w \rangle = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0. \tag{2.2}$$

The point p is said to be strongly pseudoconvex if (2.1) is positive for all $w \neq 0$ satisfying (2.2). The domain Ω is said to be pseudoconvex (resp. strongly pseudoconvex) if every point $p \in \partial\Omega$ is a point of pseudoconvex (resp. strongly pseudoconvex).

It is well-known [10] that one can choose a nice defining function ρ for a strongly pseudoconvex domain in $\mathbb{C}^n$. Such ρ plays an important role in the construction of integral formula. In fact, we have

Lemma 2.3 *Let Ω be a bounded strongly pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. Then there exists a defining function ρ and a positive constant C such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq C \,|\, w \,|^2 \tag{2.3}$$

for all $z \in \partial\Omega$ and $w \in \mathbb{C}^n$.

Proof. [10]. ❑

Now we begin to write down the integral representation formulas for bounded strongly pseudoconvex domain with C^3 boundary. Let Ω be a bounded strongly pseudoconvex domain with C^3 boundary. Let ρ be a defining function satisfying (2.3) and $\Omega_\delta = \{\, z \in \mathbb{C}^n \mid \rho(z) < \delta \,\}$ if $\delta > 0$. Moreover, all the proofs of the following results can be found in [6,13].

Lemma 2.4 *For some $\delta > 0$, there is a function $\Phi(z,\zeta) : \Omega_\delta \times \partial\Omega \to \mathbb{C}$ such that*

1. *For any fixed $\zeta \in \partial\Omega$ the function $\Phi(z,\zeta)$ is holomorphic in $z \in \Omega_\delta$ and $\Phi(z,\zeta) \neq 0$ for $z \in \bar{\Omega} - \{\zeta\}$.*

2. *For any fixed $z \in \Omega_\delta$ the function $\Phi(z,\zeta)$ is continuously differentiable with respect to $\zeta \in \partial\Omega$.*

Lemma 2.5 *There exist $G_j(z,\zeta) : \Omega_\delta \times \partial\Omega \to \mathbb{C}$, $j = 1, 2, \cdots, n$, which are functions holomorphic in $z \in \Omega_\delta$ for fixed $\zeta \in \partial\Omega$ and continuously differentiable in $\zeta \in \partial\Omega$ for fixed $z \in \Omega_\delta$.*

Definition 2.6 *Let Ω be a bounded open set in $\mathbb{C}^n$. A C^1 function $G(z,\zeta) = (G_1(z,\zeta), \ldots, G_n(z,\zeta)) \in \mathbb{C}^n$ defined for $z \in \Omega$ and ζ in some neighborhood of $\partial\Omega$ is called a Leray map for Ω if $\langle\, G(z,\zeta) \cdot \zeta - z \,\rangle \neq 0$ for all $(z,\zeta) \in \Omega \times \partial\Omega$, i.e.,*

$$\sum_{j=1}^{n} G_j(z,\zeta)(\zeta_j - z_j) \neq 0 \quad \forall (z,\zeta) \in \Omega \times \partial\Omega.$$

Next, we introduce an important theorem as follows, see [6,13].

Theorem 2.7 *Let $\Omega \subset\subset \mathbb{C}^n$ be an open set with piecewise C^1 boundary and let $G(z,\zeta)$ be a Leray map for Ω. Suppose that all derivatives of $G(z,\zeta)$ which are of order ≤ 2 in z and of order ≤ 1 in ζ are continuous for all $z \in \Omega$ and ζ in some* *neighborhood of* $\partial\Omega$*. Let* f *be a smooth* $(0,1)$*-form on* $\overline{\Omega}$*, say* $f(\zeta) = \sum_{j=1}^{n} f_j(\zeta) d\bar{\zeta}_j$*, with* $\bar{\partial} f = 0$*. Let*

$$u(z) = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi i)^n} \Bigg\{ \int_{\substack{\zeta \in \partial\Omega \\ 0 \le \lambda \le 1}} f(\zeta) \wedge \omega'_{\zeta,\lambda}(\eta(z,\zeta,\lambda)) \wedge \omega(\zeta) - \int_{\zeta \in \Omega} \sum_{j=1}^{n} \frac{f_j(\zeta)(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{2n}} \omega(\bar{\zeta}) \wedge \omega(\zeta) \Bigg\}$$

and

$$v(z) = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi i)^n} \int_{\zeta \in \partial\Omega} f(\zeta) \wedge \frac{\omega'(G(z,\zeta)) \wedge \omega(\zeta)}{\langle G(z,\zeta) \cdot \zeta - z \rangle^n}, \tag{2.4}$$

where

$$\omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

$$\eta(z,\zeta,\lambda) = (\eta_1(z,\zeta,\lambda), \ldots, \eta_n(z,\zeta,\lambda)),$$

$$\eta_j(z,\zeta,\lambda) = (1-\lambda) \frac{G_j(z,\zeta)}{\langle G(z,\zeta) \cdot \zeta - z \rangle} + \lambda \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}, \ \zeta \in \partial\Omega \text{ and } 0 \le \lambda \le 1,$$

$$\omega'_{\zeta,\lambda}(\eta(z,\zeta,\lambda)) = \sum_{j=1}^{n} (-1)^{j+1} \eta_j(z,\zeta,\lambda) \bigwedge_{k \ne j} d_{\zeta,\lambda} \eta_k(z,\zeta,\lambda)$$

and

$$\omega'(G(z,\zeta)) = \sum_{j=1}^{n} (-1)^{j+1} G_j(z,\zeta) \bigwedge_{k \ne j} \bar{\partial}_{z,\zeta} G_k(z,\zeta). \tag{2.5}$$

Then u *and* v *are continuous in* Ω *and we have*

$$f = v + \bar{\partial} u.$$

Actually, u *is a smooth function on* Ω*.*

Remark. u is denoted by $T_1 f$ and v is denoted by $L^G_{\partial\Omega}$ in [6].

By Theorem 2.7, we obtain the following two consequences.

Corollary 2.8 *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded open set with piecewise C^1-boundary and let $G(z,\zeta)$ be a Leray map for Ω depending holomorphically in $z \in \Omega$. Let f be a smooth $(0,1)$-form on $\overline{\Omega}$, say $f(\zeta) = \sum_{j=1}^{n} f_j(\zeta) d\bar{\zeta}_j$, with $\bar{\partial} f = 0$. Then the function u defined as in Theorem 2.7 is a smooth function on Ω and $\bar{\partial} u = f$.*

Proof. Since $G(z,\zeta)$ depends holomorphically in $z \in \Omega$, this implies,

$$\omega'(G(z,\zeta)) = \sum_{j=1}^{n} (-1)^{j+1} G_j(z,\zeta) \bigwedge_{k \neq j} \bar{\partial}_\zeta G_k(z,\zeta);$$

that is, (2.5) is a differential form of bidegree $(0, n-1)$ in ζ. Since f is a $(0,1)$-form on $\overline{\Omega}$, the integrand in (2.4) is a differential form of bidegree (n,n) in ζ, hence $v = 0$ by the definition of integration of differential forms. ❑

Corollary 2.9 *Let $\Omega \subset\subset \mathbb{C}^n$, $n > 2$, be a bounded open set with piecewise C^1-boundary and let $G(z,\zeta)$ be a Leray map for Ω which is holomorphic in $\zeta \in \partial\Omega$. Let f be a smooth $(0,1)$-form on $\overline{\Omega}$, say $f(\zeta) = \sum_{j=1}^{n} f_j(\zeta) d\bar{\zeta}_j$, with $\bar{\partial} f = 0$. Then the function u defined as in Theorem 2.7 is a smooth function on Ω and $\bar{\partial} u = f$.*

Proof. Since $G(z,\zeta)$ is holomorphic in $\zeta \in \partial\Omega$, we have,

$$\omega'(G(z,\zeta)) = \sum_{j=1}^{n} (-1)^{j+1} G_j(z,\zeta) \bigwedge_{k \neq j} \bar{\partial}_z G_k(z,\zeta);$$

i.e., (2.5) is a differential form of bidegree $(0,0)$ in ζ. Further, since f is a $(0,1)$-form on $\overline{\Omega}$, the integrand in (2.4) is a differential form of bidegree $(n,1)$ in ζ, hence $v = 0$ by the definition of integration of differential forms. ❑

3 Integral Representation of Solution for $\bar{\partial} u = f$ on Balls in $\mathbb{C}^n$

Given a ball

$$B_R = \{\, z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} |\, z_j\,|^2 - R^2 < 0 \,\},$$

where $R > 0$. Let

$$\rho(z) = \sum_{j=1}^{n} |\, z_j\,|^2 - R^2.$$

Clearly, ρ is a smooth defining function for B_R and B_R is a bounded open set with smooth boundary.

Lemma 3.1 *Let*

$$G(z,\zeta) = (\frac{\partial \rho}{\partial z_1}(\zeta), \frac{\partial \rho}{\partial z_2}(\zeta), \dots, \frac{\partial \rho}{\partial z_n}(\zeta)) = (\bar{\zeta}_1, \bar{\zeta}_2, \dots, \bar{\zeta}_n)$$

and

$$\Phi(z,\zeta) = \langle\, G(z,\zeta) \cdot \zeta - z \,\rangle = \sum_{j=1}^{n} \bar{\zeta}_j (\zeta_j - z_j).$$

Then

(a) *G is a Leray map for B_R.*

(b) *$G(z,\zeta)$ is holomorphic in z.*

(c) *On $\overline{B}_R \times \overline{B}_R$, $\Phi(z,\zeta) = 0 \Leftrightarrow z = \zeta$.*

(d) $2\mathrm{Re}\,\Phi(z,\zeta) \geq |\,\zeta - z\,|^2$ *holds on $B_R \times \partial B_R$, where* $\mathrm{Re}\, z$ *represents the real part of z.*

Proof. (a) If $\langle\, G(z,\zeta) \cdot \zeta - z \,\rangle = \sum_{j=1}^{n} \bar{\zeta}_j (\zeta_j - z_j) = 0$ for some $(z,\zeta) \in B_R \times \partial B_R$, then, by the Cauchy-Schwarz inequality, we obtain $R^2 = \sum_{j=1}^{n} |\,\zeta_j\,|^2 = \sum_{j=1}^{n} \bar{\zeta}_j z_j =$

$\left| \sum_{j=1}^{n} \bar{\zeta}_j z_j \right| \leq (\sum_{j=1}^{n} |\bar{\zeta}_j|^2)^{1/2} (\sum_{j=1}^{n} |z_j|^2)^{1/2} = R(\sum_{j=1}^{n} |z_j|^2)^{1/2}$. This is a contradiction because $R > (\sum_{j=1}^{n} |z_j|^2)^{1/2}$ for all $z \in B_R$.

(b) is trivial and (c) follows from (a).

(d) It follows from a simple calculation.

$$\begin{aligned}
2\mathrm{Re}\,\Phi(z,\zeta) &= \Phi(z,\zeta) + \overline{\Phi(z,\zeta)} \\
&= 2\sum_{j=1}^{n} |\zeta_j|^2 - \sum_{j=1}^{n} 2\mathrm{Re}\,(z_j\bar{\zeta}_j) \\
&\geq \sum_{j=1}^{n} |\zeta_j|^2 - \sum_{j=1}^{n} 2\mathrm{Re}\,(z_j\bar{\zeta}_j) + \sum_{j=1}^{n} |z_j|^2 \\
&= |\zeta - z|^2 > 0 \quad \text{on } B_R \times \partial B_R.
\end{aligned}$$

❑

An application of Corollary 2.8 gives

Theorem 3.2 *Given a smooth* $(0,1)$*-form* $f(z) = \sum_{j=1}^{n} f_j(z) d\bar{z}_j$ *on* $\overline{B}_R$ *with* $\bar{\partial} f = 0$ *in* $\bar{B}_R$. *Define*

$$u(z) = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi i)^n} \left\{ \int_{\partial B_R \times [0,1]} f(\zeta) \wedge \omega'(\eta) \wedge \omega(\zeta) - \int_{B_R} \sum_{j=1}^{n} \frac{f_j(\zeta)(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{2n}} \omega(\bar{\zeta}) \wedge \omega(\zeta) \right\},$$

where

$$\eta_j = (1-\lambda)\frac{\bar{\zeta}_j}{\Phi(z,\zeta)} + \lambda \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}, \;\; \lambda \in [0,1]$$

and

$$\omega'(\eta) = \sum_{j=1}^{n} (-1)^{j+1} \eta_j d\eta_1 \wedge \cdots \wedge d\eta_{j-1} \wedge d\eta_{j+1} \wedge \cdots \wedge d\eta_n.$$

Then

$$\bar{\partial} u = f \ \ on \ B_R.$$

Remark. Theorem 3.2 was given in [1,14] for the case $R = 1$, i.e., the unit ball.

4 Uniform Estimate for solution $\bar{\partial}u = f$ on Balls in $\mathbb{C}^n$

Let $f(z) = \sum_{j=1}^{n} f_j(z)d\bar{z}_j$ and $u(z)$ be as in Theorem 3.2. We will find an upper bound of the constant C explicitly such that $\|u\|_\infty \leq C\|f\|_\infty$. By Theorem 3.2, we have

$$|\, u(z) \,| \leq \frac{(n-1)!}{(2\pi)^n}(I_1 + I_2),$$

where

$$I_1 = \left| \int_{\partial B_R \times [0,1]} f(\zeta) \wedge \omega'(\eta) \wedge \omega(\zeta) \right|$$

and

$$I_2 = \left| \int_{B_R} \sum_{j=1}^{n} \frac{f_j(\zeta)(\bar{\zeta}_j - \bar{z}_j)}{|\, \zeta - z \,|^{2n}} \omega(\bar{\zeta}) \wedge \omega(\zeta) \right|.$$

First, we estimate I_2.

Lemma 4.1 *We have*

$$I_2 \leq 2^{n+1}\omega_{2n-1}R\|f\|_\infty,$$

where $\omega_{2n-1} = \dfrac{2\pi^n}{(n-1)!}$ *is the volume of the* $(2n-1)$*-dimension unit sphere in* $\mathbb{R}^{2n}$*.*

***Proof*.**

$$I_2 \leq 2^n\|f\|_\infty \int_{B(0,R)} \frac{1}{|\, \zeta - z \,|^{2n-1}} dV(\zeta)$$

$$\leq 2^n\|f\|_\infty \int_{B(z,2R)} \frac{1}{|\, \zeta - z \,|^{2n-1}} dV(\zeta)$$

$$= 2^n\|f\|_\infty \int_{\partial B(0,1)} \int_0^{2R} 1 dt d\sigma(\zeta)$$

$$= 2^{n+1}\omega_{2n-1}R\|f\|_\infty.$$

$\square$

We are ready to estimate I_1 by a series of steps. Observe that

$$I_1 \leq \|f\|_\infty \left| \int_{\partial B_R \times [0,1]} \sum_{j=1}^{n} d\bar{\zeta}_j \wedge \omega'(\eta) \wedge \omega(\zeta) \right|. \tag{4.1}$$

To calculate $\omega'(\eta) \wedge \omega(\zeta)$, we set

$$D_{\alpha_1 \ldots \alpha_{n-2}} = \det \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \\ \frac{\partial \eta_1}{\partial \lambda} & \frac{\partial \eta_2}{\partial \lambda} & \cdots & \frac{\partial \eta_n}{\partial \lambda} \\ \frac{\partial \eta_1}{\partial \bar{\zeta}_{\alpha_1}} & \frac{\partial \eta_2}{\partial \bar{\zeta}_{\alpha_1}} & \cdots & \frac{\partial \eta_n}{\partial \bar{\zeta}_{\alpha_1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \eta_1}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \frac{\partial \eta_2}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \cdots & \frac{\partial \eta_n}{\partial \bar{\zeta}_{\alpha_{n-2}}} \end{pmatrix}_{n \times n},$$

$$A_j = \det \begin{pmatrix} \frac{\partial \eta_1}{\partial \lambda} & \cdots & \frac{\partial \eta_{j-1}}{\partial \lambda} & \frac{\partial \eta_{j+1}}{\partial \lambda} & \cdots & \frac{\partial \eta_n}{\partial \lambda} \\ \frac{\partial \eta_1}{\partial \bar{\zeta}_{\alpha_1}} & \cdots & \frac{\partial \eta_{j-1}}{\partial \bar{\zeta}_{\alpha_1}} & \frac{\partial \eta_{j+1}}{\partial \bar{\zeta}_{\alpha_1}} & \cdots & \frac{\partial \eta_n}{\partial \bar{\zeta}_{\alpha_1}} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \eta_1}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \cdots & \frac{\partial \eta_{j-1}}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \frac{\partial \eta_{j+1}}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \cdots & \frac{\partial \eta_n}{\partial \bar{\zeta}_{\alpha_{n-2}}} \end{pmatrix}_{(n-1) \times (n-1)}$$

and

$$A_{jl} = \det \begin{pmatrix} \frac{\partial \eta_1}{\partial \bar{\zeta}_{\alpha_1}} & \cdots & \frac{\partial \eta_{j-1}}{\partial \bar{\zeta}_{\alpha_1}} & \frac{\partial \eta_{j+1}}{\partial \bar{\zeta}_{\alpha_1}} & \cdots & \frac{\partial \eta_{l-1}}{\partial \bar{\zeta}_{\alpha_1}} & \frac{\partial \eta_{l+1}}{\partial \bar{\zeta}_{\alpha_1}} & \cdots & \frac{\partial \eta_n}{\partial \bar{\zeta}_{\alpha_2}} \\ \frac{\partial \eta_1}{\partial \bar{\zeta}_{\alpha_2}} & \cdots & \frac{\partial \eta_{j-1}}{\partial \bar{\zeta}_{\alpha_2}} & \frac{\partial \eta_{j+1}}{\partial \bar{\zeta}_{\alpha_2}} & \cdots & \frac{\partial \eta_{l-1}}{\partial \bar{\zeta}_{\alpha_2}} & \frac{\partial \eta_{l+1}}{\partial \bar{\zeta}_{\alpha_2}} & \cdots & \frac{\partial \eta_n}{\partial \bar{\zeta}_{\alpha_2}} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \eta_1}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \cdots & \frac{\partial \eta_{j-1}}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \frac{\partial \eta_{j+1}}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \cdots & \frac{\partial \eta_{l-1}}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \frac{\partial \eta_{l+1}}{\partial \bar{\zeta}_{\alpha_{n-2}}} & \cdots & \frac{\partial \eta_n}{\partial \bar{\zeta}_{\alpha_{n-2}}} \end{pmatrix}_{(n-2) \times (n-2)},$$

where $\alpha_1, \ldots, \alpha_{n-2} \in \{1, 2, \ldots, n\}$ are distinct. Then,

$$\begin{aligned} &\omega'(\eta) \wedge \omega(\zeta) \\ &= \sum_{j=1}^{n} (-1)^{j+1} \eta_j d\eta_1 \wedge \cdots \wedge \widehat{d\eta_j} \wedge \cdots \wedge d\eta_n \wedge \omega(\zeta) \\ &= \sum_{j=1}^{n} (-1)^{j+1} \eta_j \Big(\frac{\partial \eta_1}{\partial \lambda} d\lambda + \sum_{k_1=1}^{n} \frac{\partial \eta_1}{\partial \bar{\zeta}_{k_1}} d\bar{\zeta}_{k_1}\Big) \wedge \cdots \wedge \widehat{d\eta_j} \wedge \cdots \wedge \Big(\frac{\partial \eta_n}{\partial \lambda} d\lambda + \sum_{k_n=1}^{n} \frac{\partial \eta_n}{\partial \bar{\zeta}_{k_n}} d\bar{\zeta}_{k_n}\Big) \wedge \omega(\zeta) \end{aligned}$$

$$
= \sum_{j=1}^{n} (-1)^{j+1} \eta_j \Bigg[\sum_{l<j} (-1)^{l-1} \frac{\partial \eta_l}{\partial \lambda} d\lambda \wedge (\sum_{k_1=1}^{n} \frac{\partial \eta_1}{\partial \bar{\zeta}_{k_1}} d\bar{\zeta}_{k_1}) \wedge \cdots \wedge \widehat{(\sum_{k_l=1}^{n} \frac{\partial \eta_l}{\partial \bar{\zeta}_{k_l}} d\bar{\zeta}_{k_l})} \wedge \cdots \wedge
$$

$$
\widehat{(\sum_{k_j=1}^{n} \frac{\partial \eta_j}{\partial \bar{\zeta}_{k_j}} d\bar{\zeta}_{k_j})} \wedge \cdots \wedge (\sum_{k_n=1}^{n} \frac{\partial \eta_n}{\partial \bar{\zeta}_{k_n}} d\bar{\zeta}_{k_n}) + \sum_{l>j} (-1)^{l-2} \frac{\partial \eta_l}{\partial \lambda} d\lambda \wedge (\sum_{k_1=1}^{n} \frac{\partial \eta_1}{\partial \bar{\zeta}_{k_1}} d\bar{\zeta}_{k_1}) \wedge \cdots \wedge
$$

$$
\widehat{(\sum_{k_j=1}^{n} \frac{\partial \eta_j}{\partial \bar{\zeta}_{k_j}} d\bar{\zeta}_{k_j})} \wedge \cdots \wedge \widehat{(\sum_{k_l=1}^{n} \frac{\partial \eta_l}{\partial \bar{\zeta}_{k_l}} d\bar{\zeta}_{k_l})} \wedge \cdots \wedge (\sum_{k_n=1}^{n} \frac{\partial \eta_n}{\partial \bar{\zeta}_{k_n}} d\bar{\zeta}_{k_n}) \Bigg] \wedge \omega(\zeta) + (\text{no } d\lambda \text{ terms})
$$

$$
= \sum_{j=1}^{n} (-1)^{j+1} \eta_j \Bigg[\sum_{l<j} (-1)^{l-1} \frac{\partial \eta_l}{\partial \lambda} d\lambda \wedge (\sum_{\substack{\alpha_1, \dots, \alpha_{n-2} \\ \in \{1,2,\dots,n\} \\ \textit{they are distinct}}} A_{lj} d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}})
$$

$$
+ \sum_{l>j} (-1)^{l-2} \frac{\partial \eta_l}{\partial \lambda} d\lambda \wedge (\sum_{\substack{\alpha_1, \dots, \alpha_{n-2} \\ \in \{1,2,\dots,n\} \\ \textit{they are distinct}}} A_{jl} d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}}) \Bigg] \wedge \omega(\zeta) + (\text{no } d\lambda \text{ terms})
$$

$$
= \sum_{j=1}^{n} (-1)^{j+1} \eta_j \sum_{\substack{\alpha_1, \dots, \alpha_{n-2} \\ \in \{1,2,\dots,n\} \\ \textit{they are distinct}}} \Bigg\{ \sum_{l<j} (-1)^{l+1} \frac{\partial \eta_l}{\partial \lambda} A_{lj} d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}}
$$

$$
+ \sum_{l>j} (-1)^{l+1} \left[(-1) \frac{\partial \eta_l}{\partial \lambda} \right] A_{jl} d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}} \Bigg\} \wedge \omega(\zeta) + (\text{no } d\lambda \text{ terms})
$$

$$
= \sum_{\substack{\alpha_1, \dots, \alpha_{n-2} \\ \in \{1,2,\dots,n\} \\ \textit{they are distinct}}} \left(\sum_{j=1}^{n} (-1)^{j+1} \eta_j A_j \right) d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}} \wedge \omega(\zeta) + (\text{no } d\lambda \text{ terms})
$$

$$
= \sum_{\substack{\alpha_1, \dots, \alpha_{n-2} \\ \in \{1,2,\dots,n\} \\ \textit{they are distinct}}} D_{\alpha_1 \dots \alpha_{n-2}} d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}} \wedge \omega(\zeta) + (\text{no } d\lambda \text{ terms}),
$$

where $\widehat{(\quad)}$ represents that the term is deleted. Therefore, we have proved the following Lemma.

Lemma 4.2 $\omega'(\eta) \wedge \omega(\zeta)$

$$= \sum_{\substack{\alpha_1,\ldots,\alpha_{n-2} \\ \in\{1,2,\ldots,n\} \\ \textit{they are distinct}}} D_{\alpha_1\ldots\alpha_{n-2}} d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}} \wedge \omega(\zeta) + (\text{no } d\lambda \text{ terms}),$$

where

$$D_{\alpha_1\ldots\alpha_{n-2}} = \det \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \\ \frac{\partial\eta_1}{\partial\lambda} & \frac{\partial\eta_2}{\partial\lambda} & \cdots & \frac{\partial\eta_n}{\partial\lambda} \\ \frac{\partial\eta_1}{\partial\bar{\zeta}_{\alpha_1}} & \frac{\partial\eta_2}{\partial\bar{\zeta}_{\alpha_1}} & \cdots & \frac{\partial\eta_n}{\partial\bar{\zeta}_{\alpha_1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\eta_1}{\partial\bar{\zeta}_{\alpha_{n-2}}} & \frac{\partial\eta_2}{\partial\bar{\zeta}_{\alpha_{n-2}}} & \cdots & \frac{\partial\eta_n}{\partial\bar{\zeta}_{\alpha_{n-2}}} \end{pmatrix}. \tag{4.2}$$

Next, we start to deal with $\sup_{\alpha_1,\ldots,\alpha_{n-2}} \left| D_{\alpha_1\ldots\alpha_{n-2}} \right|$. By writing down all components of the matrix of (4.2), we have, for $j = 1, 2, \ldots, n$ and $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{n-2}$,

$$\eta_j = (1-\lambda)\frac{\bar{\zeta}_j}{\Phi(z,\zeta)} + \lambda\frac{\bar{\zeta}_j - \bar{z}_j}{|\,\zeta - z\,|^2},$$

$$\frac{\partial\eta_j}{\partial\lambda} = \frac{\bar{\zeta}_j - \bar{z}_j}{|\,\zeta - z\,|^2} - \frac{\bar{\zeta}_j}{\Phi(z,\zeta)}$$

and

$$\frac{\partial\eta_j}{\partial\bar{\zeta}_\alpha} = \delta^j_\alpha\left(\frac{\lambda}{|\,\zeta - z\,|^2} + \frac{1-\lambda}{\Phi(z,\zeta)}\right)$$

$$-\lambda\frac{(\bar{\zeta}_j - \bar{z}_j)(\zeta_\alpha - z_\alpha)}{|\,\zeta - z\,|^4} - (1-\lambda)\frac{\bar{\zeta}_j(\zeta_\alpha - z_\alpha)}{\Phi(z,\zeta)^2},$$

where δ^j_k is the Kronecker delta; that is,

$$\delta^j_k = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

Applying the properties of the determinants to (4.2); i.e., adding $(1-\lambda)$ times the second row to the first and subtracting the first row from the second. We get

$$D_{\alpha_1\ldots\alpha_{n-2}} = \det\begin{pmatrix} \frac{\bar\zeta_1-\bar z_1}{|\zeta-z|^2} & \frac{\bar\zeta_2-\bar z_2}{|\zeta-z|^2} & \cdots & \frac{\bar\zeta_n-\bar z_n}{|\zeta-z|^2} \\ -\frac{\bar\zeta_1}{\Phi(z,\zeta)} & -\frac{\bar\zeta_2}{\Phi(z,\zeta)} & \cdots & -\frac{\bar\zeta_n}{\Phi(z,\zeta)} \\ \frac{\partial\eta_1}{\partial\bar\zeta_{\alpha_1}} & \frac{\partial\eta_2}{\partial\bar\zeta_{\alpha_1}} & \cdots & \frac{\partial\eta_n}{\partial\bar\zeta_{\alpha_1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\eta_1}{\partial\zeta_{\alpha_{n-2}}} & \frac{\partial\eta_2}{\partial\zeta_{\alpha_{n-2}}} & \cdots & \frac{\partial\eta_n}{\partial\zeta_{\alpha_{n-2}}} \end{pmatrix}.$$

Further, let

$$\mu = \frac{\lambda}{|\zeta-z|^2} + \frac{1-\lambda}{\Phi(z,\zeta)},$$

then we have

$$D_{\alpha_1\ldots\alpha_{n-2}} = \det\begin{pmatrix} \frac{\bar\zeta_1-\bar z_1}{|\zeta-z|^2} & \frac{\bar\zeta_2-\bar z_2}{|\zeta-z|^2} & \cdots & \frac{\bar\zeta_n-\bar z_n}{|\zeta-z|^2} \\ -\frac{\bar\zeta_1}{\Phi(z,\zeta)} & -\frac{\bar\zeta_2}{\Phi(z,\zeta)} & \cdots & -\frac{\bar\zeta_n}{\Phi(z,\zeta)} \\ \mu\delta^1_{\alpha_1} & \mu\delta^2_{\alpha_1} & \cdots & \mu\delta^n_{\alpha_1} \\ \vdots & \vdots & & \vdots \\ \mu\delta^1_{\alpha_{n-2}} & \mu\delta^2_{\alpha_{n-2}} & \cdots & \mu\delta^n_{\alpha_{n-2}} \end{pmatrix}. \tag{4.3}$$

In fact, the expression (4.3) is produced by adding $\lambda\frac{\zeta_{\alpha_j}-z_{\alpha_j}}{|\zeta-z|^2}$ times the first row to the $(2+j)$-th and subtracting $(1-\lambda)\frac{\zeta_{\alpha_j}-z_{\alpha_j}}{\Phi(z,\zeta)}$ times the second from the $(2+j)$-th, where $j=1,2,\ldots,n-2$.

In order to find an upper bound of the absolute value of determinant of a matrix over $\mathbb{C}$, we recall the Hadamard's inequality.

Lemma 4.3 *Let* $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, *where* $a_j = (a_{j1}\cdots a_{jn})$ *is the* j*-th row of* A. *Then* $|\det A| \le |a_1||a_2|\cdots|a_n|$, *where* $|a_j|^2 = \sum_{k=1}^{n}|a_{jk}|^2$ *and* $j=1,2,\ldots,n$.

Proof. It follows from definition of determinant and Cauchy-Schwarz inequality.

$$
\begin{aligned}
|\det A|^2 &= \left| \sum_{\sigma \in S_n} (\mathrm{sgn}\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \right|^2 \\
&\leq \left(\sum_{\sigma \in S_n} |a_{1\sigma(1)}| \, |a_{2\sigma(2)}| \cdots |a_{n\sigma(n)}| \right)^2 \\
&\leq \left(\sum_{k=1}^{n} |a_{1k}|^2 \right) \left(\sum_{k=1}^{n} |a_{2k}|^2 \right) \cdots \left(\sum_{k=1}^{n} |a_{nk}|^2 \right) \\
&= |a_1|^2 |a_2|^2 \cdots |a_n|^2,
\end{aligned}
$$

where S_n is the symmetric group of degree n. ❑

Let

$$
D_{\alpha_1 \ldots \alpha_{n-2}} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}.
$$

Then

$$
|d_1| = \left(\sum_{j=1}^{n} \frac{|\bar{\zeta}_j - \bar{z}_j|^2}{|\zeta - z|^4} \right)^{\frac{1}{2}} = \frac{1}{|\zeta - z|},
$$

$$
|d_2| = \left(\sum_{j=1}^{n} \frac{|\bar{\zeta}_j|^2}{|\Phi(z,\zeta)|^2} \right)^{\frac{1}{2}} = \frac{|\zeta|}{|\Phi(z,\zeta)|} = \frac{R}{|\Phi(z,\zeta)|}
$$

and for $j \geq 3$,

$$
|d_j| = |\mu| \leq \frac{\lambda}{|\zeta - z|^2} + \frac{1-\lambda}{|\Phi(z,\zeta)|} \leq \frac{2}{|\zeta - z|^2},
$$

So

$$
|D_{\alpha_1 \ldots \alpha_{n-2}}| \leq \frac{2^{n-2} R}{|\zeta - z|^{2n-3} |\Phi(z,\zeta)|}.
$$

This yields

$$\sup_{\alpha_1,\ldots,\alpha_{n-2}} \left| D_{\alpha_1\ldots\alpha_{n-2}} \right| \leq \frac{2^{n-2}R}{|\,\zeta - z\,|^{2n-3}\,|\,\Phi(z,\zeta)\,|}.$$

Now we rewrite (4.1) as follows:

$$I_1 \leq \|f\|_\infty \left| \int_{\partial B_R \times [0,1]} \left(\sum_{j=1}^{n} d\bar{\zeta}_j \right) \wedge \sum_{\substack{\alpha_1,\ldots,\alpha_{n-2} \\ \in \{1,2,\ldots,n\} \\ \textit{they are distinct}}} D_{\alpha_1\ldots\alpha_{n-2}} d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}} \wedge \omega(\zeta) \right|$$

$$= \|f\|_\infty \left| \int_{\partial B_R \times [0,1]} \sum_{j=1}^{n} \sum_{\substack{\alpha_1,\ldots,\alpha_{n-2} \\ \in \{1,\ldots,\hat{j},\ldots,n\} \\ \textit{they are distinct}}} D_{\alpha_1\ldots\alpha_{n-2}} d\bar{\zeta}_j \wedge d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}} \wedge \omega(\zeta) \right|$$

$$= (n-2)!\|f\|_\infty \left| \int_{\partial B_R \times [0,1]} \sum_{j=1}^{n} \sum_{\alpha_1 < \cdots < \alpha_{n-2}} \left| D_{\alpha_1\ldots\alpha_{n-2}} \right| d\bar{\zeta}_j \wedge d\lambda \wedge d\bar{\zeta}_{\alpha_1} \wedge \cdots \wedge d\bar{\zeta}_{\alpha_{n-2}} \wedge \omega(\zeta) \right|$$

$$\leq 2^{2n-2} n! R \|f\|_\infty \int_{\partial B_R} \frac{1}{|\,\zeta - z\,|^{2n-3}\,|\,\Phi(z,\zeta)\,|} d\sigma(\zeta).$$

Set

$$J(z) = \int_{\partial B_R} \frac{1}{|\,\zeta - z\,|^{2n-3}\,|\,\Phi(z,\zeta)\,|} d\sigma(\zeta).$$

Then $J(z)$ satisfies the following property.

Lemma 4.4 *$J(z)$ is invariant under unitary transformation of $\mathbb{C}^n$.*

Proof. Let U be a unitary transformation and let U^* be the adjoint of U. By the definition of a unitary operator and the properties of inner product, we have

$$J(U(z)) = \int_{\partial B_R} \frac{1}{|\,\zeta - U(z)\,|^{2n-3}\,|\,\Phi(U(z),\zeta)\,|} d\sigma(\zeta)$$

$$= \int_{\partial B_R} \frac{1}{|\,\zeta - U(z)\,|^{2n-3}\,|\,\langle\,\zeta - U(z), \zeta\,\rangle\,|} d\sigma(\zeta)$$

$$= \int_{\partial B_R} \frac{1}{|\, U^*(\zeta) - z \,|^{2n-3} \,|\, \langle\, U(U^*(\zeta) - z), \zeta \,\rangle \,|} d\sigma(\zeta)$$

$$= \int_{\partial B_R} \frac{1}{|\, U^*(\zeta) - z \,|^{2n-3} \,|\, \langle\, U^*(\zeta) - z, U^*(\zeta) \,\rangle \,|} d\sigma(\zeta)$$

$$= \int_{\partial B_R} \frac{1}{|\, U^*(\zeta) - z \,|^{2n-3} \,|\, \Phi(z, U^*(\zeta)) \,|} d\sigma(\zeta)$$

$$= \int_{U^*(\partial B_R)} \frac{1}{|\, \xi - z \,|^{2n-3} \,|\, \Phi(z, \xi) \,|} \,|\, \det U \,|\, d\sigma(\xi)$$

$$= \int_{\partial B_R} \frac{1}{|\, \zeta - z \,|^{2n-3} \,|\, \Phi(z, \zeta) \,|} d\sigma(\zeta)$$

$$= J(z).$$

This is the desired result. ❑

From now on we write $\zeta_j = x_j + iy_j$ for $j = 1, 2, \ldots, n$. By Lemma 4.4, we can further assume that $z = (a, 0, \ldots, 0) = q \in \mathbb{R}^{2n}$, $0 \le a < R$. We rewrite

$$J(s) = \int_{\partial B_R} \frac{1}{|\, s - q \,|^{2n-3} \,|\, \Phi(q, s) \,|} d\sigma(s)$$

$$= I_{11} + I_{12},$$

where

$$s = (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n},$$

$$\partial B_R^+ = \{\, s = (x_1, y_1, \ldots, x_n, y_n) \in \partial B_R \mid x_1 > 0 \,\},$$

$$\partial B_R^- = \{\, s = (x_1, y_1, \ldots, x_n, y_n) \in \partial B_R \mid x_1 \le 0 \,\},$$

$$I_{11} = \int_{\partial B_R^+} \frac{1}{|\, s - q \,|^{2n-3} \,|\, \Phi(q, s) \,|} d\sigma(s)$$

and

$$I_{12} = \int_{\partial B_R^-} \frac{1}{\mid s-q \mid^{2n-3} \mid \Phi(q,s) \mid} d\sigma(s).$$

Our goal now is to evaluate I_{11} and I_{12}.

Lemma 4.5 *We have*

$$I_{12} \le \frac{\omega_{2n-1}}{2}.$$

Proof. Clearly, I_{12} is less than the integral

$$\int_{\substack{\mid s \mid^2 = R^2 \\ x_1 \le 0}} \frac{d\sigma(s)}{[(x_1-a)^2 + y_1^2 + \sum_{j=2}^n (x_j^2+y_j^2)]^{n-\frac{3}{2}} \left| \sum_{j=1}^n (x_j^2+y_j^2) - ax_1 + iay_1 \right|},$$

where $s = (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$. Take the sphere coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{2n-2})$, where $0 \le r < R$, $0 \le \theta_1 \le \pi$ and $0 \le \theta_j \le 2\pi$ for all $2 \le j \le 2n-2$. More precisely,

$$\begin{cases} r^2 = y_1^2 + \sum_{j=2}^n (x_j^2 + y_j^2) \\ y_1 = r\sin\theta_1 \\ x_2 = r\cos\theta_1 \sin\theta_2 \\ \qquad \vdots \\ x_n = r\cos\theta_1 \cdots \cos\theta_{2n-3}\sin\theta_{2n-2} \\ y_n = r\cos\theta_1 \cdots \cos\theta_{2n-3}\cos\theta_{2n-2}. \end{cases}$$

Since

$$(x_1-a)^2 + y_1^2 + \sum_{j=2}^n (x_j^2+y_j^2) = (x_1-a)^2 + r^2 \ge r^2$$

and

$$\left| \sum_{j=1}^n (x_j^2+y_j^2) - ax_1 + iay_1 \right| = \left| R^2 + a\sqrt{R^2-r^2} + iar\sin\theta_1 \right| \ge R^2,$$

we have

$$I_{12} \leq \omega_{2n-1} \int_0^R \frac{r^{2n-2}}{r^{2n-3}R^2} dr$$

$$= \frac{\omega_{2n-1}}{2}.$$

❑

Finally, we estimate I_{11}. Let

$$t_1(s) = \rho(s) - \rho(q),$$

and for $j = 2, 3, \ldots, 2n$,

$$t_j(s) = \begin{cases} y_{\frac{j}{2}} & \text{if } j \text{ is even} \\ \\ x_{\frac{j+1}{2}} & \text{if } j \text{ is odd,} \end{cases}$$

then

(a) $\nabla t_1(s) = \nabla \rho(s) = 2s$, in particular, $|\nabla t_1(s)| = 2R$ for all $s \in \partial B_R^+$.

(b) $\sum_{j=2}^{2n} t_j^2(s) = R^2 - x_1^2 \leq R^2$ for all $s \in \partial B_R^+$.

(c) $\det(\nabla t_1(s), \ldots, \nabla t_{2n}(s)) = \frac{\partial \rho(s)}{\partial x_1} = 2x_1 > 0$ for all $s \in \partial B_R^+$.

Lemma 4.6 $d\sigma(s) = \frac{1}{4R}(R^2 - \sum_{j=2}^{2n} t_j^2)^{-\frac{1}{2}} dt_2 \wedge \cdots \wedge dt_{2n}$.

Proof. We know that $dt_1 \wedge \cdots \wedge dt_{2n} = \mathrm{Jac}(t) dx_1 \wedge \cdots \wedge dy_n$, where

$$\mathrm{Jac}(t) = \det \begin{pmatrix} \frac{\partial t_1(s)}{\partial x_1} & \frac{\partial t_1(s)}{\partial y_1} & \cdots & \frac{\partial t_1(s)}{\partial y_n} \\ \frac{\partial t_2(s)}{\partial x_1} & \frac{\partial t_2(s)}{\partial y_1} & \cdots & \frac{\partial t_2(s)}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial t_{2n}(s)}{\partial x_1} & \frac{\partial t_{2n}(s)}{\partial y_1} & \cdots & \frac{\partial t_{2n}(s)}{\partial y_n} \end{pmatrix}.$$

By above arguments (a)-(c) and definition of $d\sigma(s)$, we get

$$\mathrm{Jac}(t) = \frac{\partial t_1(s)}{\partial x_1} = 2x_1 = 2\sqrt{R^2 - \sum_{j=2}^{2n} t_j^2}$$

and

$$\begin{aligned} d\sigma(s) &= \frac{\nabla\rho(s)}{|\nabla\rho(s)|} \rfloor dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= \frac{\nabla t_1(s)}{|\nabla t_1(s)|} \rfloor dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= \frac{\nabla t_1}{|\nabla t_1|} \rfloor \frac{1}{\mathrm{Jac}(t)} dt_1 \wedge \cdots \wedge dt_{2n} \\ &= \frac{1}{\mathrm{Jac}(t)\,|\nabla t_1|} \sum_{j=1}^{2n} (-1)^{j+1} \frac{\partial t_1}{\partial t_j} dt_1 \wedge \cdots \wedge dt_{j-1} \wedge dt_{j+1} \wedge \cdots \wedge dt_{2n} \\ &= \frac{1}{\mathrm{Jac}(t)\,|\nabla t_1|} dt_2 \wedge \cdots \wedge dt_{2n} \\ &= \frac{1}{4R} (R^2 - \sum_{j=2}^{2n} t_j^2)^{-\frac{1}{2}} dt_2 \wedge \cdots \wedge dt_{2n}, \end{aligned}$$

where $\rfloor$ denotes the interior multiplication of the vector and the differential form.

❑

Lemma 4.7 *Given* $a \in [0, R]$, *we have* $R^2 - a^2 + 2ar \geq Rr \quad \forall r \in [0, R]$.

Proof. Divide the problem into two parts.

Case 1: If $a \leq r$, then $R^2 - a^2 + 2ar \geq R^2 + ar \geq R^2 \geq Rr$.

Case 2: If $a > r$, then

$$\begin{aligned} R^2 - a^2 + 2ar &= (R + a)(R - a) + 2ar \\ &\geq (R + r)(R - a) + 2ar \\ &= Rr + R^2 - Ra + ar \\ &\geq Rr. \end{aligned}$$

The proof is completed. ❑

On the other hand, since $\Phi(q,s) = R^2 - ax_1 + iay_1$, we have

(1) $\operatorname{Re}\Phi(q,s) = R^2 - ax_1$

$$
\begin{aligned}
&= \frac{1}{2}\left[2R^2 + (x_1 - a)^2 - x_1^2 - a^2\right] \\
&\geq \frac{1}{2}\left[2R^2 - x_1^2 - a^2\right]
\end{aligned}
$$

(2) $\operatorname{Im}\Phi(q,s) = ay_1$,

where $\operatorname{Im} z$ represents the imaginary part of z. Applying Lemma 4.7, we obtain

$$
\begin{aligned}
|\,\Phi(q,s)\,| &\geq \frac{1}{2}(|\operatorname{Re}\Phi(q,s)\,| + |\operatorname{Im}\Phi(q,s)\,|) \\
&\geq \frac{1}{4}\left(2R^2 - x_1^2 - a^2 + 2a\,|\,y_1\,|\right) \\
&= \frac{1}{4}\left\{(R^2 - a^2 + 2a\,|\,y_1\,|) + \left[y_1^2 + \sum_{j=2}^{n}(x_j^2 + y_j^2)\right]\right\} \\
&\geq \frac{1}{4}\left[R\,|\,y_1\,| + y_1^2 + \sum_{k=2}^{n}(x_j^2 + y_j^2)\right].
\end{aligned}
$$

Using Lemma 4.6 and the spherical coordinates again, we have

$$
\begin{aligned}
I_{11} &= \int_{\partial B_R^+} \frac{1}{|\,s - q\,|^{2n-3}\,|\,\Phi(q,s)\,|}\,d\sigma(s) \\
&\leq \int_{\sum_{j=2}^{2n} t_j^2 \leq R^2} \frac{1}{(\sum_{j=2}^{2n} t_j^2)^{n-\frac{3}{2}}\frac{1}{4}(R\,|\,t_1\,| + \sum_{j=2}^{2n} t_j^2)} \cdot \frac{1}{4R\sqrt{R^2 - \sum_{j=2}^{2n} t_j^2}}\,dt_2 \wedge \cdots \wedge dt_{2n} \\
&\leq \frac{1}{R}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{r^{2n-2}}{r^{2n-3}r(R\sin\theta_1 + r)} \cdot \frac{1}{\sqrt{R^2 - r^2}}\,dr\,d\theta_1\cdots d\theta_{2n-2} \\
&= \frac{(2\pi)^{2n-3}}{R^3}\int_0^{\pi}\int_0^{R} \frac{1}{(\sin\theta_1 + \frac{r}{R})} \cdot \frac{1}{\sqrt{1 - (\frac{r}{R})^2}}\,dr\,d\theta_1
\end{aligned}
$$

$$= \frac{(2\pi)^{2n-3}}{R^2} \int_0^1 \left(\int_0^\pi \frac{1}{\sin\theta_1 + r} d\theta_1 \right) \frac{1}{\sqrt{1-r^2}} dr.$$

By the residue theorem,

$$\int_0^\pi \frac{1}{\sin\theta_1 + r} d\theta_1 \ \leq \ \frac{1}{2} \int_0^{2\pi} \frac{1}{\sin^2\theta_1 + r} d\theta_1 = \frac{\pi}{\sqrt{r}\sqrt{r+1}}.$$

Therefore we obtain

$$\begin{aligned} I_{11} &\leq \frac{2^{2n-3}\pi^{2n-2}}{R^2} \int_0^1 \frac{1}{\sqrt{r}\sqrt{r+1}} \cdot \frac{1}{\sqrt{1-r^2}} dr \\ &\leq \frac{2^{2n-3}\pi^{2n-2}}{R^2} \int_0^1 \frac{1}{\sqrt{r}} \cdot \frac{1}{\sqrt{1-r}} dr \\ &= \frac{2^{2n-2}\pi^{2n-2}}{R^2} \int_0^1 \frac{1}{\sqrt{1-u^2}} du \\ &= \frac{2^{2n-2}\pi^{2n-2}}{R^2} (\arcsin u \Big|_0^1) \\ &= \frac{2^{2n-3}\pi^{2n-1}}{R^2} \end{aligned}$$

and we have the following lemmas.

Lemma 4.8 $I_{11} \leq \dfrac{2^{2n-3}\pi^{2n-1}}{R^2}.$

Lemma 4.9 $I_1 \ \leq \ 2^{2n-2} n! R (\dfrac{\omega_{2n-1}}{2} + \dfrac{2^{2n-3}\pi^{2n-1}}{R^2}) \|f\|_\infty.$

So far we can summarize all the results obtained.

1. $|\, u(z) \,| \ \leq \ \dfrac{(n-1)!}{(2\pi)^n} (I_1 + I_2).$

2. $I_2 \ \leq \ 2^{n+1} \omega_{2n-1} R \|f\|_\infty.$

3. $I_1 \ \leq \ 2^{2n-2} n! R \|f\|_\infty \displaystyle\int_{\partial B_R} \frac{1}{|\, \zeta - z \,|^{2n-3} \,|\, \Phi(z,\zeta) \,|} d\sigma(\zeta).$

4. $J(s) = \int_{\partial B_R} \frac{1}{|\zeta - z|^{2n-3} |\Phi(z,\zeta)|} d\sigma(\zeta) = I_{11} + I_{12}.$

5. $I_{11} = \int_{\partial B_R^+} \frac{1}{|s - q|^{2n-3} |\Phi(q,s)|} d\sigma(s).$

6. $I_{12} = \int_{\partial B_R^-} \frac{1}{|s - q|^{2n-3} |\Phi(q,s)|} d\sigma(s).$

7. $I_{12} \le \frac{\omega_{2n-1}}{2}.$

8. $I_{11} \le \frac{2^{2n-3}\pi^{2n-1}}{R^2}.$

9. $I_1 \le 2^{2n-2} n! R(\frac{\omega_{2n-1}}{2} + \frac{2^{2n-3}\pi^{2n-1}}{R^2})\|f\|_\infty.$

10.
$$|u(z)| \le \frac{(n-1)!}{(2\pi)^n}\left\{2^{2n-2} n! R(\frac{\omega_{2n-1}}{2} + \frac{2^{2n-3}\pi^{2n-1}}{R^2})\|f\|_\infty + 2^{n+1}\omega_{2n-1} R\|f\|_\infty\right\}$$
$$= \{2^{n-2} n! R + 2^{3n-5}\pi^{n-1}(n-1)! n! R^{-1} + 4R\}\|f\|_\infty,$$

where we use the formula $\omega_{2n-1} = \frac{2\pi^n}{(n-1)!}$.

Theorem 4.10 *Given a smooth* $(0,1)$*-form* $f(z) = \sum_{j=1}^{n} f_j(z) d\bar{z}_j$ *on* $\bar{B}_R$ *with* $\bar{\partial} f = 0$ *in* B_R*. Let* u *be the function defined in Theorem* 3.2*. Then we have* $\bar{\partial} u = f$ *on* B_R *and* $\|u\|_\infty \le C_{B_R}\|f\|_\infty$*, where*

$$C_{B_R} = 2^{n-2} n! R + 2^{3n-5}\pi^{n-1}(n-1)! n! R^{-1} + 4R.$$

With $R = 1$, we have the following consequence.

Corollary 4.11 *Given a smooth closed* $(0,1)$*-form* $f(z) = \sum_{j=1}^{n} f_j(z) d\bar{z}_j$ *on the unit ball* $\bar{B}$*. Let* u *be the function defined in Theorem* 3.2*. Then we have* $\bar{\partial} u = f$ *on* B *and* $\|u\|_\infty \le C_B\|f\|_\infty$*, where*

$$C_B = 2^{n-2} n! + 2^{3n-5}\pi^{n-1}(n-1)! n! + 4.$$

Remark. This corollary was also obtained in [1,14] with different constants. However, our estimate is more accurate and precise. Moreover, in the estimate of [14], there are some gaps.

5 Uniform Estimate of Solution for $\bar{\partial}u = f$ on Shell Domains in $\mathbb{C}^n$

Given a spherical shell domain

$$S = S(R_1, R_2) = \{\, z \in \mathbb{C}^n \mid R_1^2 < \sum_{j=1}^{n} |\, z_j \,|^2 < R_2^2 \,\},$$

where $0 < R_1 < R_2$. Let

$$\rho_1(z) = \sum_{j=1}^{n} |\, z_j \,|^2 - R_2^2 \text{ and } \rho_2(z) = R_1^2 - \sum_{j=1}^{n} |\, z_j \,|^2.$$

Then $S = \{\, z \in \mathbb{C}^n \mid \rho_j(z) < 0,\ j = 1, 2 \,\}$ and it is a smooth bounded domain in $\mathbb{C}^n$. Denote $\partial S = S_1 \cup S_2$, where $S_j = \{\, z \in \mathbb{C}^n \mid \sum_{j=1}^{n} |\, z_j \,|^2 = R_j^2 \,\}$ for $j = 1, 2$. Take

$$\tilde{G}(z,\zeta) = \begin{cases} \bar{\zeta} \ = (\bar{\zeta}_1, \dots, \bar{\zeta}_n) & \text{on } S \times S_2 \\ -\bar{z} = (-\bar{z}_1, \dots, -\bar{z}_n) & \text{on } S \times S_1 \end{cases}$$

and

$$\Psi(z,\zeta) = \begin{cases} \langle\, \bar{\zeta} \cdot \zeta - z \,\rangle \ = \sum_{j=1}^{n} \bar{\zeta}_j(\zeta_j - z_j) & \text{for all } (z,\zeta) \in S \times S_2 \\ \langle\, -\bar{z} \cdot \zeta - z \,\rangle = \sum_{j=1}^{n} (-\bar{z}_j)(\zeta_j - z_j) & \text{for all } (z,\zeta) \in S \times S_1. \end{cases}$$

Lemma 5.1 *With $\tilde{G}$ as defined above, we have*

(a) *$\tilde{G}$ is a Leray map for S.*

(b) *On $S \times S_2$, $\tilde{G}(z,\zeta)$ is holomorphic in z.*

(c) *On $S \times S_1$, $\tilde{G}(z,\zeta)$ is holomorphic in ζ.*

(d) $2\mathrm{Re}\,\Psi(z,\zeta) \geq |\, \zeta - z \,|^2$ *holds on $S \times \partial S$.*

Proof. (a) By Lemma 3.1, it remains to check that $\tilde{G}(z,\zeta)$ satisfies the definition of a Leray map on $(z,\zeta) \in S \times S_1$. Seeking a contradiction, we suppose that for some $(z,\zeta) \in S \times S_1$ such that $\Psi(z,\zeta) = 0$, we have $\sum_{j=1}^{n} \zeta_j \bar{z}_j = |z|^2$. By Cauchy Schwarz inequality again, we see that

$$|z|^2 = \left| \sum_{j=1}^{n} \zeta_j \bar{z}_j \right| \le \left(\sum_{j=1}^{n} |\zeta_j|^2 \right)^{1/2} \left(\sum_{j=1}^{n} |\bar{z}_j|^2 \right)^{1/2} = |z| R_1.$$

This contradicts the fact $|z| > R_1$ for all $z \in S$. Hence,

$$\Psi(z,\zeta) \neq 0 \quad \text{for all } (z,\zeta) \in S \times S_1.$$

Therefore, $\tilde{G}$ is a Leray map for S.

(b) and (c) are trivial.

(d) It follows from a similar calculation with Lemma 3.1(d). ❑

Remark. The Leray map in Lemma 5.1 was mentioned in [6].

By Corollary 2.8, Corollary 2.9 and Lemma 5.1, we have

Theorem 5.2 *Let* $f(z) = \sum_{j=1}^{n} f_j(z) d\bar{z}_j$ *be a smooth* $(0,1)$*-form on* $\overline{S}$ *with* $\bar{\partial} f = 0$ *on* S. *Define*

$$u(z) = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi i)^n} \left\{ \int_{\partial S \times [0,1]} f(\zeta) \wedge \omega'(\eta) \wedge \omega(\zeta) - \int_S \sum_{j=1}^{n} \frac{f_j(\zeta)(\bar{\zeta}_j - \bar{z}_j)}{|\zeta - z|^{2n}} \omega(\bar{\zeta}) \wedge \omega(\zeta) \right\},$$

where the j*-th component of* η *is*

$$\eta_j = (1-\lambda) \frac{\bar{\zeta}_j}{\Psi(z,\zeta)} + \lambda \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}, \quad \lambda \in [0,1].$$

Then

$\bar{\partial} u = f$ *on* S.

We begin to estimate $\|u\|_\infty \leq C_S \|f\|_\infty$. Write

$$|\,u(z)\,| \;\leq\; \frac{(n-1)!}{(2\pi)^n}(\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3),$$

where

$$\tilde{I}_1 = \left| \int_{S_2\times[0,1]} f(\zeta)\wedge\omega'(\eta)\wedge\omega(\zeta) \right|,$$

$$\tilde{I}_2 = \left| \int_{S_1\times[0,1]} f(\zeta)\wedge\omega'(\eta)\wedge\omega(\zeta) \right|$$

and

$$\tilde{I}_3 = \left| \int_S \sum_{j=1}^n \frac{f_j(\zeta)(\bar{\zeta}_j - \bar{z}_j)}{|\,\zeta - z\,|^{2n}} \omega(\bar{\zeta})\wedge\omega(\zeta) \right|.$$

Clearly, as in the arguments of section 4, we obtain

Lemma 5.3 (A) $\tilde{I}_1 \;\leq\; 2^{2n-2} n! R_2 (\frac{\omega_{2n-1}}{2} + \frac{2^{2n-3}\pi^{2n-1}}{R_2^2})\|f\|_\infty.$

(B) $\tilde{I}_3 \;\leq\; 2^{n+1}\omega_{2n-1} R_2 \|f\|_\infty.$

Now we estimate $\tilde{I}_2$. For $j = 1, 2, \ldots, n$ and $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{n-2}$,

$$\eta_j = (1-\lambda)\frac{\bar{\zeta}_j}{\Psi(z,\zeta)} + \lambda \frac{\bar{\zeta}_j - \bar{z}_j}{|\,\zeta - z\,|^2},$$

$$\frac{\partial \eta_j}{\partial \lambda} = \frac{\bar{\zeta}_j - \bar{z}_j}{|\,\zeta - z\,|^2} - \frac{\bar{\zeta}_j}{\Psi(z,\zeta)}$$

and

$$\frac{\partial \eta_j}{\partial \bar{\zeta}_\alpha} = \delta_\alpha^j \left(\frac{\lambda}{|\,\zeta - z\,|^2} + \frac{1-\lambda}{\Psi(z,\zeta)} \right) - \lambda \frac{(\bar{\zeta}_j - \bar{z}_j)(\zeta_\alpha - z_\alpha)}{|\,\zeta - z\,|^4}.$$

Via the same procedure in Page 11-16, we get

$$\tilde{I}_2 \;\leq\; 2^{2n-2} n! R_1 \|f\|_\infty \int_{S_1} \frac{1}{|\,\zeta - z\,|^{2n-3}\,|\,\Psi(z,\zeta)\,|} d\sigma(\zeta).$$

Hence it is reduced to estimate the integral

$$\tilde{J}(z) = \int_{S_1} \frac{1}{|\,\zeta - z\,|^{2n-3}\,|\,\Psi(z,\zeta)\,|} d\sigma(\zeta).$$

Lemma 5.4 *$\tilde{J}(z)$ is invariant under unitary transformation of $\mathbb{C}^n$.*

Proof. Let U be a unitary transformation of $\mathbb{C}^n$ and U^* be its adjoint. We have

$$\begin{aligned}
\tilde{J}(U(z)) &= \int_{S_1} \frac{1}{|\,\zeta - U(z)\,|^{2n-3}\,|\,\Psi(U(z),\zeta)\,|} d\sigma(\zeta) \\
&= \int_{S_1} \frac{1}{|\,\zeta - U(z)\,|^{2n-3}\,|\,\langle\,\zeta - U(z), -U(z)\,\rangle\,|} d\sigma(\zeta) \\
&= \int_{S_1} \frac{1}{|\,U^*(\zeta) - z\,|^{2n-3}\,|\,\langle\,U^*(\zeta) - z, -z\,\rangle\,|} d\sigma(\zeta) \\
&= \int_{S_1} \frac{1}{|\,U^*(\zeta) - z\,|^{2n-3}\,|\,\Psi(z, U^*(\zeta))\,|} d\sigma(\zeta) \\
&= \int_{U^*(S_1)} \frac{1}{|\,\xi - z\,|^{2n-3}\,|\,\Psi(z,\xi)\,|}\,|\det U\,|\, d\sigma(\xi) \\
&= \int_{S_1} \frac{1}{|\,\zeta - z\,|^{2n-3}\,|\,\Psi(z,\zeta)\,|} d\sigma(\zeta) \\
&= \tilde{J}(z).
\end{aligned}$$

Hence, $\tilde{J}(z)$ is invariant under unitary transformation of $\mathbb{C}^n$. ❑

Hence, we may assume that $z = (a, 0, \ldots, 0) = q \in \mathbb{R}^{2n}$, where $R_1 < a < R_2$. Let $\zeta_j = x_j + iy_j$ for $j = 1, 2, \ldots, n$, we can rewrite

$$\begin{aligned}
\tilde{J}(s) &= \int_{S_1} \frac{1}{|\,s - q\,|^{2n-3}\,|\,\Psi(q,s)\,|} d\sigma(s) \\
&= \tilde{I}_{21} + \tilde{I}_{22},
\end{aligned}$$

where

$$s = (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n},$$

$$S_1^+ = \{\, s = (x_1, y_1, \dots, x_n, y_n) \in S_1 \mid x_1 \; > \; 0 \,\},$$

$$S_1^- = \{\, s = (x_1, y_1, \dots, x_n, y_n) \in S_1 \mid x_1 \; \leq \; 0 \,\},$$

$$\tilde{I}_{21} = \int_{S_1^+} \frac{1}{\mid s - q \mid^{2n-3} \mid \Psi(q, s) \mid} d\sigma(s)$$

and

$$\tilde{I}_{22} = \int_{S_1^-} \frac{1}{\mid s - q \mid^{2n-3} \mid \Psi(q, s) \mid} d\sigma(s).$$

Lemma 5.5 $\tilde{I}_{22} \leq \dfrac{\omega_{2n-1}}{2}$.

Proof. Clearly,

$$\tilde{I}_{22} \leq \int_{\substack{\mid s \mid^2 = R_1^2 \\ x_1 \leq 0}} \frac{d\sigma(s)}{[(x_1 - a)^2 + y_1^2 + \sum_{j=2}^{n} (x_j^2 + y_j^2)]^{n - \frac{3}{2}} \mid a^2 - a x_1 - i a y_1 \mid},$$

where $s = (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$ and $R_1 < a < R_2$. Take the sphere coordinates as follows:

$$\begin{cases} r^2 = y_1^2 + \displaystyle\sum_{j=2}^{n} (x_j^2 + y_j^2) \\ y_1 = r \sin\theta_1 \\ x_2 = r \cos\theta_1 \sin\theta_2 \\ \qquad \vdots \\ x_n = r \cos\theta_1 \cdots \cos\theta_{2n-3} \sin\theta_{2n-2} \\ y_n = r \cos\theta_1 \cdots \cos\theta_{2n-3} \cos\theta_{2n-2}, \end{cases}$$

where $0 \leq r < R$, $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_j \leq 2\pi$ for all $2 \leq j \leq 2n-2$. Since $x_1 = -\sqrt{R_1^2 - r^2}$, we have

$$|\, a^2 - ax_1 - iay_1 \,| = \left| a^2 + a\sqrt{R_1^2 - r^2} - iar\sin\theta_1 \right| \geq a^2 + a\sqrt{R_1^2 - r^2} \geq a^2 \geq R_1^2.$$

Hence, $\tilde{I}_{22} \leq \omega_{2n-1} \int_0^{R_1} \frac{r^{2n-2}}{r^{2n-3} R_1^2} dr = \frac{\omega_{2n-1}}{2}.$

❑

The last work to do is to estimate $\tilde{I}_{21}$. Since $\Psi(q,s) = a^2 - ax_1 - iay_1$,

(1) $$\begin{aligned} \operatorname{Re}\Psi(q,s) &= a^2 - ax_1 \\ &= \frac{1}{2}\left\{ a^2 - R_1^2 + \left[(x_1 - a)^2 + y_1^2 + \sum_{j=2}^{n} (x_j^2 + y_j^2) \right] \right\} \\ &\geq \frac{1}{2}\left[y_1^2 + \sum_{j=2}^{n} (x_j^2 + y_j^2) \right]. \end{aligned}$$

(2) $\operatorname{Im}\Psi(q,s) = -ay_1$.

Therefore,

$$\begin{aligned} |\, \Psi(q,s) \,| &\geq \frac{1}{2}(|\operatorname{Re}\Psi(q,s)| + |\operatorname{Im}\Psi(q,s)|) \\ &\geq \frac{1}{4}\left[y_1^2 + \sum_{k=2}^{n} (x_j^2 + y_j^2) + 2R_1 \,|\, y_1 \,| \right]. \end{aligned}$$

On S_1, let

$$\tilde{t}_1(s) = \rho_2(q) - \rho_2(s),$$

and for $j = 2, 3, \ldots, 2n$,

$$\tilde{t}_j(s) = \begin{cases} y_{\frac{j}{2}} & \text{if } j \text{ is even} \\ \\ x_{\frac{j+1}{2}} & \text{if } j \text{ is odd,} \end{cases}$$

then

(a) $\nabla\tilde{t}_1(s) = -\nabla\rho_2(s) = 2s$, in particular, $\left|\nabla\tilde{t}_1(s)\right| = 2R_1$ for all $s \in S_1$.

(b) $\sum_{j=2}^{2n} \tilde{t}_j^2(s) = R_1^2 - x_1^2 \leq R_1^2$ for all $s \in S_1$.

(c) $\det(\nabla \tilde{t}_1(s), \ldots, \nabla \tilde{t}_{2n}(s)) = -\dfrac{\partial \rho_2(s)}{\partial x_1} = 2x_1 > 0$ for all $s \in S_1$.

By mean of the similar way in the Lemma 4.6, we have the following result.

Lemma 5.6 $d\sigma(s) = \dfrac{1}{4R_1}(R_1^2 - \sum_{j=2}^{2n} \tilde{t}_j^2)^{-1/2} d\tilde{t}_2 \wedge \cdots \wedge d\tilde{t}_{2n}$.

So we obtain

$$\tilde{I}_{21} \leq \int_{\sum_{j=2}^{2n} \tilde{t}_j^2 < R_1^2} \frac{4}{\left(\sum_{j=2}^{2n} \tilde{t}_j^2\right)^{n-\frac{3}{2}} (2R_1 \left| \tilde{t}_2 \right| + \sum_{j=2}^{2n} \tilde{t}_j^2)} \cdot \frac{1}{4R_1(R_1^2 - \sum_{j=2}^{2n} \tilde{t}_j^2)^{1/2}} d\tilde{t}_2 \wedge \cdots \wedge d\tilde{t}_{2n}$$

$$\leq \frac{1}{R_1} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^{\pi} \int_0^{R_1} \frac{r^{2n-2}}{r^{2n-3}(2R_1 r \sin\theta_1 + r^2)} \cdot \frac{1}{\sqrt{R_1^2 - r^2}} dr d\theta_1 \cdots d\theta_{2n-2}$$

$$\leq \frac{2^{2n-3}\pi^{2n-3}}{R_1} \int_0^{\pi} \int_0^{R_1} \frac{1}{(2R_1 \sin\theta_1 + r)} \cdot \frac{1}{\sqrt{R_1^2 - r^2}} dr d\theta_1$$

$$\leq \frac{2^{2n-4}\pi^{2n-3}}{R_1^3} \int_0^{\pi} \int_0^{R_1} \frac{1}{(\sin\theta_1 + r/2R_1)} \cdot \frac{1}{\sqrt{1 - r/R_1}} dr d\theta_1$$

$$= \frac{2^{2n-4}\pi^{2n-3}}{R_1^3} \int_0^{R_1} \left(\int_0^{\pi} \frac{1}{\sin\theta_1 + r/2R_1} d\theta_1 \right) \frac{1}{\sqrt{1 - r/R_1}} dr$$

$$\leq \frac{2^{2n-4}\pi^{2n-2}}{R_1^3} \int_0^{R_1} \frac{1}{\sqrt{r/2R_1}} \frac{1}{\sqrt{1 - r/R_1}} dr$$

$$\leq \frac{2^{2n-\frac{7}{2}}\pi^{2n-2}}{R_1^2} \int_0^{R_1} \frac{1}{\sqrt{r}\sqrt{R_1 - r}} dr$$

$$= \frac{2^{2n-\frac{7}{2}}\pi^{2n-1}}{R_1^2}$$

and we derive the following lemmas.

Lemma 5.7 $\tilde{I}_{21} \leq \dfrac{2^{2n-\frac{7}{2}}\pi^{2n-1}}{R_1^2}$.

Lemma 5.8 $\tilde{I}_2 \;\leq\; 2^{2n-2}n!R_1(\dfrac{2^{2n-\frac{7}{2}}\pi^{2n-1}}{R_1^2} + \dfrac{\omega_{2n-1}}{2})\|f\|_\infty$.

Finally we summarize all the result obtained.

1. $\mid u(z) \mid \;\leq\; \dfrac{(n-1)!}{(2\pi)^n}(\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3)$.

2. $\tilde{I}_1 \;\leq\; 2^{2n-2}n!R_2(\dfrac{\omega_{2n-1}}{2} + \dfrac{2^{2n-3}\pi^{2n-1}}{R_2^2})\|f\|_\infty$.

3. $\tilde{I}_3 \;\leq\; 2^{n+1}\omega_{2n-1}R_2\|f\|_\infty$.

4. $\tilde{I}_2 \;\leq\; 2^{2n-2}n!\|f\|_\infty \displaystyle\int_{S_1} \frac{1}{\mid \zeta - z \mid^{2n-3} \mid \Psi(z,\zeta) \mid} d\sigma(\zeta)$.

5. $\tilde{J}(z) = \displaystyle\int_{S_1} \frac{1}{\mid \zeta - z \mid^{2n-3} \mid \Psi(z,\zeta) \mid} d\sigma(\zeta) = \tilde{I}_{21} + \tilde{I}_{22}$.

6. $\tilde{I}_{21} = \displaystyle\int_{S_1^+} \frac{1}{\mid s - q \mid^{2n-3} \mid \Psi(q,s) \mid} d\sigma(s)$.

7. $\tilde{I}_{22} = \displaystyle\int_{S_1^-} \frac{1}{\mid s - q \mid^{2n-3} \mid \Psi(q,s) \mid} d\sigma(s)$.

8. $\tilde{I}_{22} \;\leq\; \dfrac{\omega_{2n-1}}{2}$.

9. $\tilde{I}_{21} \leq \dfrac{2^{2n-\frac{7}{2}}\pi^{2n-1}}{R_1^2}$.

10. $\tilde{I}_2 \;\leq\; 2^{2n-2}n!R_1(\dfrac{\omega_{2n-1}}{2} + \dfrac{2^{2n-\frac{7}{2}}\pi^{2n-1}}{R_1^2})\|f\|_\infty$.

11. $\mid u(z) \mid \;\leq\; \dfrac{(n-1)!}{(2\pi)^n} \Bigg\{ 2^{2n-2}n!R_2(\dfrac{\omega_{2n-1}}{2} + \dfrac{2^{2n-3}\pi^{2n-1}}{R_2^2}) + 2^{2n-2}n!R_1(\dfrac{\omega_{2n-1}}{2} + \dfrac{2^{2n-\frac{7}{2}}\pi^{2n-1}}{R_1^2})$

$$+2^{n+1}\omega_{2n-1}R_2\Bigg\}\|f\|_\infty$$

$$\leq \left\{2^{3n-5}\pi^{n-1}(n-1)!n!R_2^{-1} + 2^{n-2}n!R_2 + 4R_2 + 2^{n-2}n!R_1 + 2^{3n-\frac{11}{2}}\pi^{n-1}(n-1)!n!R_1^{-2}\right\}\|f\|_\infty,$$

where we use the formula $\omega_{2n-1} = \dfrac{2\pi^n}{(n-1)!}$.

Therefore, we obtain our main result in this thesis as follows.

Theorem 5.9 *Let* $f(z) = \sum_{j=1}^{n} f_j(z)d\bar{z}_j$ *be a smooth* $(0, 1)$*-form on* $\overline{S}$ *with* $\bar{\partial}f = 0$. *Let* u *be the function defined in Theorem* 5.2. *Then* $\bar{\partial}u = f$ *on* S *and* $\|u\|_\infty \leq C_S\|f\|_\infty$, *where*

$$C_S = 2^{3n-5}\pi^{n-1}(n-1)!n!R_2^{-1} + 2^{n-2}n!R_2 + 4R_2 + 2^{n-2}n!R_1 + 2^{3n-\frac{11}{2}}\pi^{n-1}(n-1)!n!R_1^{-2}.$$