2 Solved By Recurrence Relation

2.1 Solution by Recurrence Relation for Question 1

If there are k different points in an 1-dimensional line, then what is the largest number of the different regions which are formed by these k different points?

We solve this question by recurrence relation as follows:

Let a_k be the largest number of the different regions which are formed by these k different lines.

Suppose we have drawn $k-1$ points in a line and these $k-1$ points have created the largest number of different regions. Then the k^{th} point must separate one of them into two regions to create the largest number of regions. This makes one more region than the former one (see Figure 1).

Figure 1: Translation of Recurrence Relation about Point and Line

Hence, we get a recurrence relation

$$a_k = a_{k-1} + 1$$

Using iteration we have the following result

$$a_k = a_{k-1} + 1 = (a_{k-2} + 1) + 1 = \cdots = (a_0 + 1) + 1 + \cdots + 1 = 1 + k = C_0^k + C_1^k$$

2.2 Solution by Recurrence Relation for Question 2

If there are k different 1-dimensional lines in an 2-dimensional plane, then what is the largest number of the different regions which are formed by these k different lines?

This question is solved by recurrence relation as follows:

Let b_k be the largest number of the different regions which are formed by these k different lines.

Suppose we have drawn $k-1$ lines in a plane and these $k-1$ lines have created the largest number of different regions. Then the k^{th} line must intersects each former $k-1$ lines with a point and these $k-1$ points must be distinct in order to make the largest number of different regions.

Then we focus on the k^{th} line and the $k-1$ points. The original regions formed by the $k-1$ lines do not eliminate after the k^{th} line is drawn, and then each of the more regions created by the k^{th} line will be connected to each region which is created by the k^{th} line and the $k-1$ points (see Figure 2). That is the number of the more regions which could be viewed as a_{k-1} in Question 1.

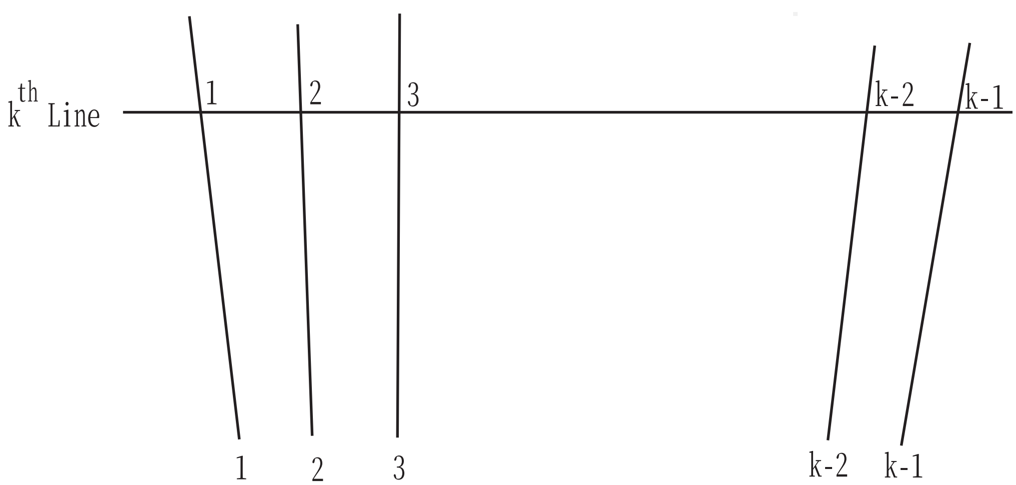

Figure 2: Translation of Recurrence Relation about Line and Plane

Hence, we get a recurrence relation

$$b_k = b_{k-1} + a_{k-1}$$

And then, we put $a_{k-1} = C_0^{k-1} + C_1^{k-1}$ into the equation to get

$$b_k = b_{k-1} + C_0^{k-1} + C_1^{k-1}$$

So, we have

$$\begin{array}{rccccccc}
 & b_k & = & b_{k-1} & + & C_0^{k-1} & + & C_1^{k-1} \\
 & b_{k-1} & = & b_{k-2} & + & C_0^{k-2} & + & C_1^{k-2} \\
 & b_{k-2} & = & b_{k-3} & + & C_0^{k-3} & + & C_1^{k-3} \\
 & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
 & b_3 & = & b_2 & + & C_0^2 & + & C_1^2 \\
 & b_2 & = & b_1 & + & C_0^1 & + & C_1^1 \\
+) & b_1 & = & b_0 & + & C_0^0 & + & C_1^0 \\
\hline
 & b_k & = & b_0 & + & \sum_{i=0}^{k-1} C_0^i & + & \sum_{i=0}^{k-1} C_1^i
\end{array}$$

Then we use the equality[5]

$$C_r^r + C_r^{r+1} + C_r^{r+2} + \cdots + C_r^n = C_{r+1}^{n+1} \tag{2.1}$$

We have (2.2) by setting $r = 0$, $n = k-1$ and (2.3) by setting $r = 1$, $n = k-1$ in Equation 2.1 in the following:

$$C_0^0 + C_0^1 + C_0^2 + \cdots + C_0^{k-1} = C_1^k \tag{2.2}$$

and

$$C_1^0 + C_1^1 + C_1^2 + \cdots + C_1^{k-1} = C_2^k \tag{2.3}$$

At last, we have the result since $b_0 = 1 = C_0^k$

$$\begin{array}{rcccccc}
b_k & = & 1 & + & \sum_{i=0}^{k-1} C_0^i & + & \sum_{i=0}^{k-1} C_0^i \\
 & = & C_0^k & + & C_1^k & + & C_2^k
\end{array}$$

2.3 Solution by Recurrence Relation for Question 3

If there are k different 2-dimensional planes in an 3-dimensional space, then what is the largest number of the different regions which are formed by these k different planes?

This question is also solved by recurrence relation in the following:

Let c_k be the largest number of the different regions which are formed by these k different planes.

Suppose we have drawn $k-1$ planes in a space and these $k-1$ planes have created the largest number of different regions. Then the k^{th} planes must intersects each former $k-1$ planes with a line and these $k-1$ line must be distinct as in Question 2 for making the largest number of different regions.

Then we focus on the k^{th} plane and the $k-1$ lines. The original regions formed by the $k-1$ planes do not eliminate after the k^{th} plane is drawn, and then each of the more regions created by the k^{th} plane will be connected to each region which is created by the k^{th} plane and the $k-1$ lines (see Figure 3). That is the number of the more regions which could be viewed as b_{k-1} in Question 2.

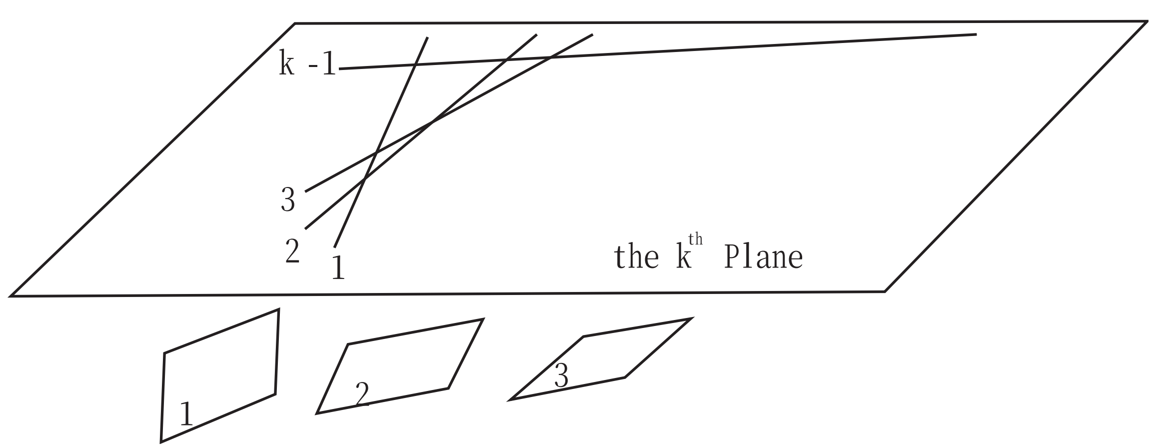

Figure 3: Translation of Recurrence Relation about Plane and Space

Hence, we get a recurrence relation

$$c_k = c_{k-1} + b_{k-1}$$

And then, we put $b_{k-1} = C_0^{k-1} + C_1^{k-1} + C_2^{k-1}$ into the equation to get

$$c_k = c_{k-1} + C_0^{k-1} + C_1^{k-1} + C_2^{k-1}$$

So, we have

$$\begin{array}{rccccccccccc}
 & c_k & = & c_{k-1} & + & C_0^{k-1} & + & C_1^{k-1} & + & C_2^{k-1} \\
 & c_{k-1} & = & c_{k-2} & + & C_0^{k-2} & + & C_1^{k-2} & + & C_2^{k-2} \\
 & c_{k-2} & = & c_{k-3} & + & C_0^{k-3} & + & C_1^{k-3} & + & C_2^{k-3} \\
 & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & c_3 & = & c_2 & + & C_0^2 & + & C_1^2 & + & C_2^2 \\
 & c_2 & = & c_1 & + & C_0^1 & + & C_1^1 & + & C_2^1 \\
+) & c_1 & = & c_0 & + & C_0^0 & + & C_1^0 & + & C_2^0 \\
\hline
 & c_k & = & c_0 & + & \sum_{i=0}^{k-1} C_0^i & + & \sum_{i=0}^{k-1} C_1^i & + & \sum_{i=0}^{k-1} C_2^i
\end{array}$$

Again we use the Equation 2.1

$$C_r^r + C_r^{r+1} + C_r^{r+2} + \cdots + C_r^n = C_{r+1}^{n+1}$$

We have (2.4) by setting $r = 0$, $n = k-1$ and (2.5) by setting $r = 1$, $n = k-1$ and (2.6) by setting $r = 2$, $n = k-1$ in Equation 2.1

$$C_0^0 + C_0^1 + C_0^2 + \cdots + C_0^{k-1} = C_1^k \tag{2.4}$$

$$C_1^0 + C_1^1 + C_1^2 + \cdots + C_1^{k-1} = C_2^k \tag{2.5}$$

and

$$C_2^0 + C_2^1 + C_2^2 + \cdots + C_2^{k-1} = C_3^k \tag{2.6}$$

At last, we have the result since $c_0 = 1 = C_0^k$

$$\begin{array}{rcccccccc}
c_k & = & 1 & + & \sum_{i=0}^{k-1} C_0^i & + & \sum_{i=0}^{k-1} C_0^i & + & \sum_{i=0}^{k-1} C_2^i \\
 & = & C_0^k & + & C_1^k & + & C_2^k & + & C_3^k
\end{array}$$