Part II

Studies of Forests

Chapter 4

On Enumeration of Plane Forests

A *plane tree* is an ordered tree with a designated vertex called the root. An $n-$*plane tree* is a plane tree with n edges. A *2-ary tree* is a plane tree where each vertex has at most two children. A *full binary tree* is a plane tree where each internal vertex has exactly two children. A *short bush* is a plane tree where no vertex has only one child. A *tall bush* is a plane tree where the root has exactly one child and the other internal vertices has at least two children.

For many results of plane trees, we refer to [7, 27, 63, 64, 65, 66, 74, 79]. Among these results, several counting formulas are very useful. For example, the *Catalan number* C_n counts $n-$plane trees or full binary trees with $2n$ edges (see [75]); the *Riordan number* R_n counts short bushes with n edges (see [7], p.85); the *Motzkin number* M_n counts $2-$ary trees with n edges or plane bushes with $n+1$ edges (see [7], p.87); and the *Narayana number* $N(n, i)$ counts $n-$plane trees with exactly i leaves (see [28]).

A *Dyck path* begins at the origin, ends at the $x-$axis with rise step $u = (1, 1)$ and fall step $d = (1, -1)$, and never goes below the $x-$axis. An $n-$*Dyck path* is a Dyck path ending at $(2n, 0)$. Consider a path P from the origin to $(2n, 0)$ with rise and fall steps. We say that P is an $n-$*Dyck path with k flaws* if P has k fall steps below the $x-$axis for $k \geq 0$. A good reference for Dyck path enumeration is a published paper by E. Deutsch [25]. In fact, an $n-$Dyck path has a one to one correspondence to an $n-$plane tree or a full binary tree with $2n$ edges. It suggests that we may learn more results of plane trees through Dyck paths.

A *forest* is a graph of no cycle. A *plane forest* is a forest where each component is a plane tree. In the literature, there are many articles studying labelled forests on $[n]$ (see [13, 14, 28, 35, 56, 58, 74, 78]). However, there are few papers investigating plane forests. It is well known that the number $P(r)$ of plane forests with n vertices and k components of type r is given by

$$P(r) = \frac{k}{n}\binom{n}{r_0, r_1, ..., r_m}, \tag{4.0.1}$$

where $r = (r_0, r_1, ..., r_m) \in \mathbb{N}^{m+1}$, with $\sum r_i = n$ and $\sum(1-i)r_i = k > 0$ (see [74], Theorem 5.3.10). A typical application of this result is a proof of Lagrange Inversion Formula.

A very useful formula in this topic is

$$\frac{k}{2n-k}\binom{2n-k}{n}, \tag{4.0.2}$$

which counts three families: $n-$Dyck paths with k returns [25], $n-$plane trees where the root has k children, and plane forests with n vertices and k components.

In this chapter, our main purpose is to generalize some results on plane trees to plane forests. In section 4.1, our main result is to evaluate the number of plane forests with n edges and k vertices at level one (Theorem 4.1.1). Consequently, we yield a new Catalan identity by means of two different methods to count plane forests. Moreover, we can evaluate the number of plane forests of n edges with x_i vertices at level i for $i = 1, 2, ..., m$ (Theorem 4.1.3).

Shapiro [65] found the number of leaves is half of the number of vertices among $n-$plane trees. In section 4.2, using a bijection, we can generalize this result to plane forests (Theorem 4.2.1). In addition, we discover two facts on plane forests with n edges and k nontrivial components: if $n + k$ is odd, then the number of plane forests with an odd number of leaves is equal to that with an even number of leaves; otherwise, they have a difference relative to the Catalan number (Theorem 4.2.2). In Theorem 4.2.4, we offer a formula to count plane forests with n edges, k components, and i nontrivial leaves. In particular, it is the Narayana number $N(n, i)$ as $k = 1$.

In section 4.3, Theorem 4.3.1 presents two relations between plane forests with vertices allowing one child and plane forests without vertices having only one child to yield four well-known identities. Example 4.3.2 generalizes a Motzkin-Catalan identity and a Catalan-Riordan identity. Example 4.3.3 generalizes a Catalan-Motzkin identity and a Riordan-Catalan identity. Therefore, we will obtain two explicit formulas of $M_{n,k}$ and $R_{n,k}$ which are generalizations of the Moztkin number and the Riordan number, respectively.

An $n-$*Motzkin path* begins at the origin, ends at the $x-$axis, and never goes below the $x-$axis with n steps containing rise step u, fall step d, and level step l where $l = (1, 0)$. In section 4.4, we list six Riordan families consisting of four classes of 2-ary trees and two classes of Motzkin paths. We use bijective proof to show all results. Finally, using one of the six Riordan families, we can obtain another formula of $R_{n,k}$ (Equation 4.4.1).

4.1 A Catalan Identity

Let the *level number* of each vertex x in a plane forest be the length of path from its root to x. In this section, we focus on counting plane forests with x_i vertices at level i. Our main purpose is to investigate the children of roots and derives a Catalan identity.

For convenience, let $\mathbb{F}_n^*$ be the set of plane forests with n edges and no trivial components, $\mathbb{F}_{n,k}$ be the set of plane forests with n edges and k components, and $\mathbb{F}_{n,k}^*$ be the subset of $\mathbb{F}_n^*$ with k components. Note that each component in $\mathbb{F}_n^*$ has at least one edges but each component in $\mathbb{F}_{n,k}$ allows no edge.

Moreover, let $\mathbb{D}_n^*$ be the set of $n-$Dyck paths with flaws. We define a component of Dyck paths with flaws to be the maximal path all above (below) the $x-$ axis and denote the set of $n-$Dyck paths with flaws and k components as $\mathbb{D}_{n,k}^*$.

By equation (4.0.2), $|\mathbb{F}_{n,k}|$ equals to $\frac{k}{2n+k}\binom{2n+k}{n}$. Due to the fact that $\mathbb{F}_n^*$ ($\mathbb{F}_{n,k}^*$) has one-to-two correspondence to $\mathbb{D}_n^*$ ($\mathbb{D}_{n,k}^*$) (see Figure 5.2), it is obvious

$$|\mathbb{F}_n^*| = \frac{1}{2}\binom{2n}{n} = \frac{n+1}{2}C_n. \tag{4.1.1}$$

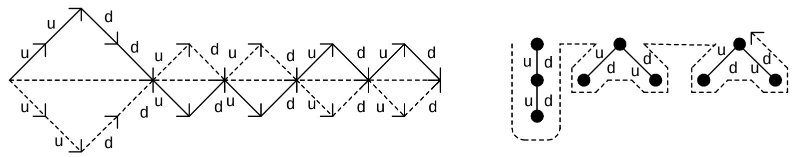

Figure 4.1: Two 6-Dyck paths with flaws symmetric with respect to the $x-$axis and their common corresponding plane forest of 6 edges

Touchard's identity [41, 57, 64, 79] tells us that the $(n+1)^{st}$ Catalan number

$$C_{n+1} = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} C_k.$$

This identity motivate us to derive another Catalan identity. By virtue of counting plane forests in $\mathbb{F}_n^*$ in two different methods, we obtain the following result.

Theorem 4.1.1 *The number of plane forests in* $\mathbb{F}_n^*$ *with* k *vertices at level one equals to* $\frac{2^{k-1}k}{2n-k}\binom{2n-k}{n}$ *and thus*

$$C_n = \frac{1}{n+1} \sum_{k=1}^{n} \frac{k2^k}{2n-k} \binom{2n-k}{n}.$$

Proof. Set $\mathbb{A}_k$ to be the subset of $\mathbb{F}_n^*$ with k vertices at level one. Let $\mathbb{B}_k$ be the set of plane forests with n vertices and k components where each vertex in each component has the same color either black or white. Since each component is 2-colored either black or white, by equation (4.0.2), $|\mathbb{B}_k| = \frac{k2^k}{2n-k}\binom{2n-k}{n}$. For each plane forest in $\mathbb{A}_k$, we alternately color all components either black or white and let it be another set $\mathbb{A}_k'$. We will use a bijection between $\mathbb{B}_k$ and $\mathbb{A}_k'$ to prove that $|\mathbb{A}_k| = \frac{1}{2}|\mathbb{A}_k'| = \frac{1}{2}|\mathbb{B}_k|$ (see Figure 4.2).

One the one hand, for a given plane forest in $\mathbb{A}_k'$, if we delete the roots, then the remaining is a plane forest in $\mathbb{B}_k$. One the other hand, for a given plane forest in $\mathbb{B}_k$, if we append a new black (white) vertex to connect roots of successive black (white) components, then there yields a plane forest in $\mathbb{A}_k'$. Hence, there is a bijection between $\mathbb{A}'$ and $\mathbb{B}_k$ and the first part is proved.

The second part, a Catalan identity, is completed if we use two methods to evaluate the cardinality of $\mathbb{F}_n^*$: One is to use equation (4.1.1) and the other is $|\mathbb{F}_n^*| = \sum_{k \geq 1} |F_{n,k}^*|$. ❑

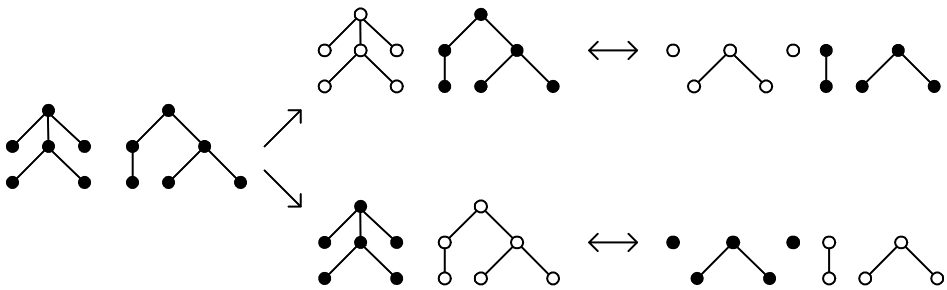

Figure 4.2: A plane forest of 10 edges with 5 vertices at level one and its two corresponding forests with 10 vertices and 5 components

Lemma 4.1.2 *The number of plane forests in $\mathbb{F}_{n,k}$, with x_i vertices at level i for $i = 1, 2, ..., m$ is*

$$\frac{x_m}{2(n-x)-x_m}\binom{2(n-x)-x_m}{n-x}\prod_{i=1}^{m}\binom{x_i+x_{i-1}-1}{x_i},$$

where $x = \sum_{i=1}^{m-1} x_i$ *and* $x_0 = k$.

Proof. We use induction on m. For $m = 1$, let f be a plane forest in $\mathbb{F}_{n,k}$ with x_1 vertices at level one. If we delete the root of each component in f, then the remaining f' is a plane forest in $\mathbb{F}_{n-x_1,x_1}$. By equation (4.0.2), $|\mathbb{F}_{n-x_1,x_1}|$ equals to $\frac{x_1}{2n-x_1}\binom{2n-x_1}{n}$. However, given $f' \in \mathbb{F}_{n-x_1,x_1}$, there are $\binom{x_1+k-1}{x_1}$ ways to enlarge f' to $f \in \mathbb{F}_{n,k}$ with x_1 vertices at level one. Therefore, there are $\frac{x_1}{2n-x_1}\binom{2n-x_1}{n}\binom{x_1+k-1}{x_1}$ plane forests in $\mathbb{F}_{n,k}$ with x_1 vertices at level one. Hence the initial step holds.

Assume that the statement is true for $m = l$. For $m = l + 1$, let f be a plane forest in $\mathbb{F}_{n,k}$ with x_i vertices at level i for $i = 1, 2, ..., l + 1$. If we delete the root of each component, then the remaining f' is a plane forest in $\mathbb{F}_{n-x_1,x_1}$ with x_{i+1} vertices at level i for $i = 1, 2, ..., l$. By inductive hypothesis, there are

$$\frac{x_{l+1}}{2(n-x_1-x')-x_{l+1}}\binom{2(n-x_1-x')-x_{l+1}}{n-x_1-x'}\prod_{i=2}^{l+1}\binom{x_i+x_{i-1}-1}{x_i}$$

plane forests in $\mathbb{F}_{n-x_1,x_1}$ with x_{i+1} vertices at level i for $i = 1, 2, ..., l$, where $x' = \sum_{i=2}^{l} x_i$.

However, given $f' \in \mathbb{F}_{n-x_1,x_1}$, there are $\binom{x_1+k-1}{x_1}$ ways to enlarge f' to $f \in \mathbb{F}_{n,k}$ with x_1 vertices at level one, i.e., there are

$$\binom{x_1+k-1}{x_1}\frac{x_{l+1}}{2(n-x)-x_{l+1}}\binom{2(n-x)-x_{l+1}}{n-x}\prod_{i=2}^{l+1}\binom{x_i+x_{i-1}-1}{x_i}$$

plane forests in $\mathbb{F}_{n,k}$ with x_i vertices at level i for $i=1,2,...,l+1$, where $x=\sum_{i=1}^{l} x_i$. Hence,we complete the proof. ❑

Appending a vertex to connected the roots in $\mathbb{F}_{n,k}$ yields a plane tree with $n+k$ edges. Hence Lemma 4.1.2 has the other explanation as follows: The number of $n-$plane trees with x_i vertices at level $i+1$ or the number of $n-$Dyck paths with x_i rise steps between $y=i-1$ and $y=i$ for $i=0,1,2,...,m$ is

$$\frac{x_m}{2(n-x)-x_m}\binom{2(n-x)-x_m}{n-x}\prod_{i=1}^{m}\binom{x_i+x_{i-1}-1}{x_i},$$

where $x=\sum_{i=0}^{m-1} x_i$.

By Theorem 4.1.1 and Lemma 4.1.2, one easily obtains the following result.

Theorem 4.1.3 *Let $x=\sum_{i=2}^{m-1} x_i$. Then,*

1. *the number of plane forests in $\mathbb{F}_n^*$, with x_i vertices at level i for $i=1,2,...,m$, is*
$$\frac{2^{x_1-1}x_m}{2(n-x)-x_m}\binom{2(n-x)-x_m}{n-x}\prod_{i=2}^{m}\binom{x_i+x_{i-1}-1}{x_i},\ \textit{and}$$
2. *the number of $n-$Dyck paths with flaws with x_i rise steps between $y=i-1$ and $y=i$ or $y=-i+1$ and $y=-i$ for $i=1,2,...,m$ is*
$$\frac{2^{x_1}x_m}{2(n-x)-x_m}\binom{2(n-x)-x_m}{n-x}\prod_{i=2}^{m}\binom{x_i+x_{i-1}-1}{x_i}.$$

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4.2 Some Results of Leaves

In this section we focus on studying the leaves of plane forests. In [66], Shapiro used generating function to show that among the vertices of $n-$plane trees, exactly half of them are leaves, offered it as a problem in [65]. Seo [63] presented an insightful bijective proof via Dyck paths.

Earlier, there are some methods which may illustrate the Shapiro's problem. One of them is Theorem 2 in [23] which leads the fact: The number of $n-$plane trees with i leaves is equal to the number of $n-$plane trees with i internal vertices. Therefore, we may generalize the result to the plane forests, whose proof is a modification of the second proof of Theorem 2 in [23].

Theorem 4.2.1 *Among the vertices of plane forests in $\mathbb{F}^*_{n,k}$, exactly half of them are leaves for $n \geq k$.*

In fact, for a given forest $f_{n,k}$ with i leaves in $\mathbb{F}^*_{n,k}$, using up/down walk, $f_{n,k}$ corresponds to a full binary forest $f'_{2n,k}$ (a forest where each component is a full binary tree) of $n+k$ leaves in $\mathbb{F}^*_{2n,k}$. Since $f_{n,k}$ has i leaves and $n+k-i$ internal vertices, $f'_{2n,k}$ has i left leaves and $n+k-i$ right leaves. If we reflect $f'_{2n,k}$, then there yields a full binary forest $g'_{2n,k}$ with $n+k-i$ left leaves and i right leaves in $\mathbb{F}^*_{2n,k}$. Similarly, $g'_{2n,k}$ corresponds to a forest $g_{n,k}$ with $n+k-i$ leaves and i internal vertices in $\mathbb{F}^*_{n,k}$.

In what follows, we show that the above technique of proof can yield more results: The following Theorem is a generalization of Theorem 4 in [33].

Theorem 4.2.2 *In $\mathbb{F}^*_{n,k}$, let $e_{n,k}$ and $o_{n,k}$ be the numbers of plane forests with even number of leaves and odd number of leaves, respectively. Then $e_{n,k} - o_{n,k} = 0$, if $n+k$ is odd; $(-1)^{\frac{n+k}{2}} \frac{k}{n} \binom{n}{\frac{n+k}{2}}$, otherwise.*

Proof. Let E_k and O_k be generating functions of plane forests with even number of leaves and odd number of leaves in $\mathbb{F}^*_{n,k}$, respectively. Clearly,

$$E_k = E_1 E_{k-1} + O_1 O_{k-1} \text{ and } O_k = E_1 O_{k-1} + O_1 E_{k-1}.$$

By recursively enumerating, $E_k - O_k = (E_1 - O_1)^k$. In [23], p. 21, Theorem 4, we know that there are $\frac{k}{2i-k}\binom{2i-k}{i}$ $i-$plane trees with root having degree k , i.e.

$$|\mathbb{F}_{i-k,k}| = \frac{k}{2i-k}\binom{2i-k}{i}.$$

Combining this with

$$e_{2i,1} - o_{2i,1} = 0$$

and

$$e_{2i+1,1} - o_{2i+1,1} = (-1)^{i+1}C_i \ ([33],\ \text{p. } 194,\ \text{Theorem } 4),$$

we obtain

$$e_{n,k} - o_{n,k} = 0,\ \text{if } n+k \text{ is odd}$$

and

$$e_{n,k} - o_{n,k} = (-1)^{\frac{n+k}{2}}\frac{k}{n}\binom{n}{\frac{n+k}{2}},\ \text{otherwise.}$$

❏

Remark. In case of even $n+k$, it is curious whether there is another combinatorial proof. In fact, for odd $n+k$, we can give a bijective proof as follows: Given a forest f in $\mathbb{F}^*_{n,k}$ with even (odd) number of leaves, as above, f corresponds to a forest g in $\mathbb{F}^*_{n,k}$ with even (odd) number of internal vertices, i.e. g has odd (even) number of leaves due to $n+k$ is odd.

Figure 4.3 is an example with $n = 5$ and $k = 2$ to illustrate Theorem 4.2.1 and Remark; compare Figure 4.3 with Fig. 3. on p. 13 of [23].

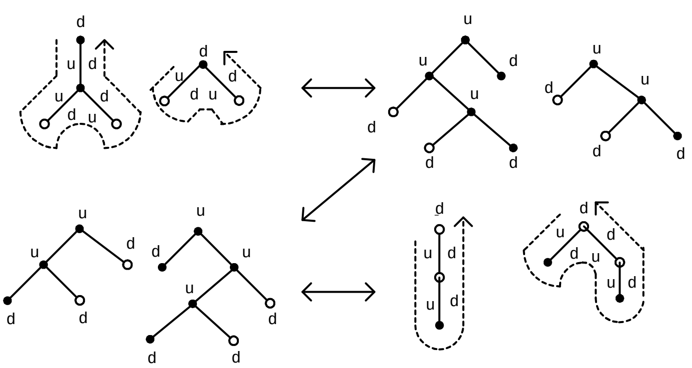

Figure 4.3: A plane forest with 4 leaves and its corresponding plane forest with 4 internal vertices.

As a direct consequence of Theorem 4.2.2, we obtain:

Corollary 4.2.3 *Among the peaks and valleys of $n-$Dyck paths with flaws and k components,*

1. *if $n+k$ is odd, then the number of Dyck paths with an even number of peaks and valleys equals to that with an odd number of peaks and valleys;*
2. *if $n+k$ is even, then the difference between the number of Dyck paths with an even number of peaks and valleys and that with an odd number of peaks and valleys is $(-1)^{\frac{n+k}{2}}\frac{2k}{n}\binom{n}{\frac{n+k}{2}}$.*

Deutsch ([25], p. 174), by virtue of generating function, showed that the number of $n-$Dyck paths with i peak is the Narayana number $\frac{1}{n}\binom{n}{i}\binom{n}{i-1}$, and it is also equal to the number of $n-$plane trees with i leaves (see [23], Theorem 1). In what follows, we use a bijection analogous to the proof of Theorem 5.3.10 in [74] to generalize the result to the case of plane forests. Note that the proof is an immediate consequence of Lemma 4.7.12 in [73].

Theorem 4.2.4 *The number of plane forests with $i(\geq k)$ nontrivial leaves in $\mathbb{F}_{n,k}$ is*

$$\frac{k}{n}\binom{n}{i}\binom{n+k-i}{i-1}.$$

In particular, if $k=1$, then there are $\frac{1}{n}\binom{n}{i}\binom{n}{i-1}$ $n-$plane trees of i leaves.

Proof. Let $\mathbb{F}_{n,k,i}$ be the set of forests in $\mathbb{F}_{n,k}$ with i nontrivial leaves. Let $\mathbb{W}_{n,k,i}$ be the set of sequences of length $2n+k$ with n u's, $n+k$ d's and starting with u such that all d's follow from exact u. Clearly,

$$|\mathbb{W}_{n,k,i}| = \binom{n}{i}\binom{n+k-i}{i-1}.$$

Define a map $\psi : \mathbb{F}_{n,k,i} \times [n] \to \mathbb{W}_{n,k,i} \times [k]$ as following (see Figure 4.4). Let $\tau \in \mathbb{F}_{n,k,i}$ and $w \in \mathbb{W}_{n,k,i}$ such that w is the bijective correspondence of τ. Then set $\psi(\tau, j) = (w_j, l)$, where w_j is a cyclic of w starting with the j^{th} u and the j^{th} u in preorder appears at l^{th} component of τ.

To define $\psi^{-1} : \mathbb{W}_{n,k,i} \times [k] \to \mathbb{F}_{n,k,i} \times [n]$, we proceed the following process. For $(w,l) \in \mathbb{W}_{n,k,i} \times [k]$, we choose $w_j \in \mathbb{W}_{n,k,i}$ and $\tau \in \mathbb{F}_{n,k,i}$ satisfy the following conditions:

1. τ is the correspondence of w_j;

2. w_j is a cyclic of w starting with the j^{th} u; and

3. the j^{th} u in preorder appears at l^{th} component of τ.

Then we set $\psi^{-1}(w, l) = (\tau, j)$.

From the bijection, we obtain $n|\mathbb{F}_{n,k,i}| = k|\mathbb{W}_{n,k,i}|$, and we complete the proof.
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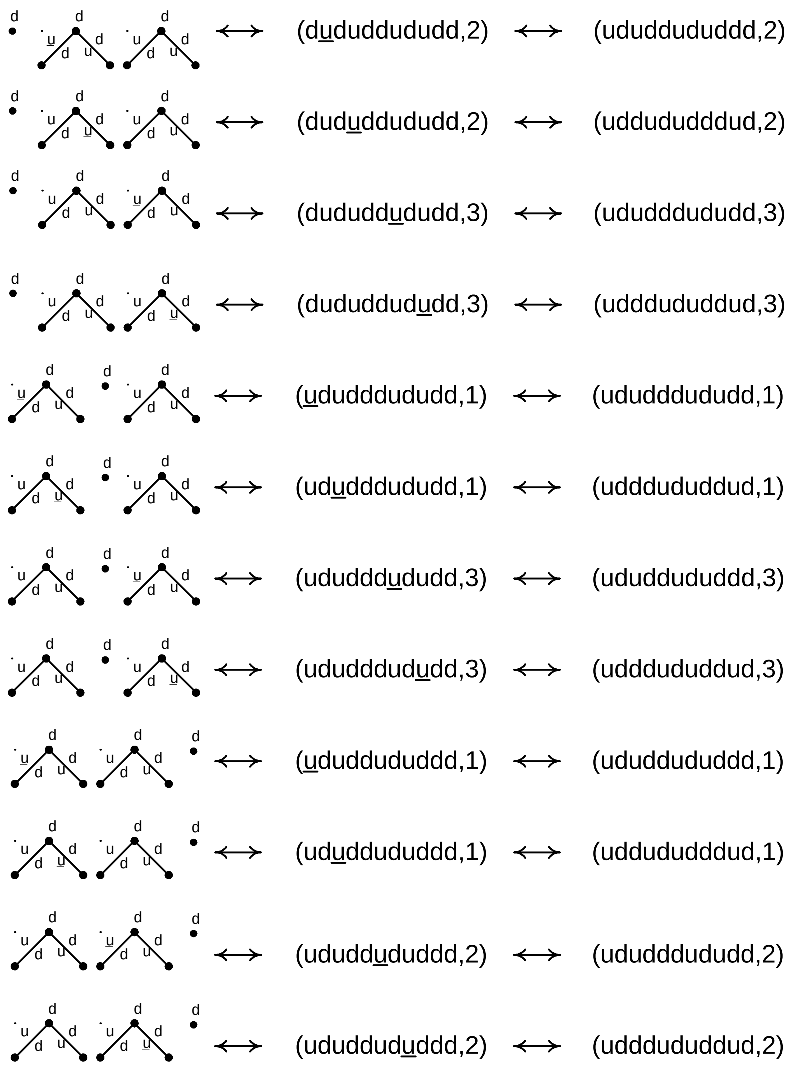

Figure 4.4: An illustration for the proof of Theorem 4.2.4.

4.3 Generalizations of Motzkin-Catalan Identity

Bernhart ([7], p. 99), using linear operator, presented six identities including four Motzkin-Catalan identities and two Catalan-Riordan identities. We herein generalize four among these six identities.

For convenience, let $\mathbb{F}_{n,k}(\vec{s})$ be the subset of $\mathbb{F}_{n,k}$ such that every internal vertex of each forest in $\mathbb{F}_{n,k}(\vec{s})$ has s_i children for some $i = 1, 2, ..., m$, where $\vec{s} = (s_1, s_2, ..., s_m) \in \mathbb{N}^m$ and $1 \leq s_1 < s_2 < \cdots < s_m \leq n$. In particular, let $C_{n,k}$ count plane forests $\mathbb{F}_{n,k}$ or full binary forests $\mathbb{F}_{2n,k}(2)$, $M_{n,k}$ count plane forests $\mathbb{F}_{n,k}(1,2)$, and $R_{n,k}$ count plane forests $\mathbb{F}_{n,k}(2, 3, ..., n)$.

The following identities between $|\mathbb{F}_{n,k}(1,\vec{s})|$ and $|\mathbb{F}_{n,k}(\vec{s})|$ are generalizations of two of the four Motzkin-Catalan identity and the two Catalan-Riordan identity obtained in [7]. However, our technique doesn't seem to yield analogous generalizations of the rest two identities.

Theorem 4.3.1 *Let $\vec{s}$ be a vector with each coordinate a positive integer greater than one. Then*

1. $|\mathbb{F}_{n,k}(1,\vec{s})| = \sum_{i\geq 0} \binom{n+k-1}{n-i} |\mathbb{F}_{i,k}(\vec{s})|$ *and*

2. $|\mathbb{F}_{n,k}(\vec{s})| = \sum_{i\geq 0} (-1)^{n-i} \binom{n+k-1}{n-i} |\mathbb{F}_{i,k}(1,\vec{s})|$.

Proof.

1. Given a forest in $\mathbb{F}_{n,k}(1,\vec{s})$, suppose that the forest has j vertices of exactly one child. Then there are $|\mathbb{F}_{n-j,k}(\vec{s})|$ plane forests with no vertex allowing one child and there are $n+k-j$ positions, k positions of which are above each root, to place j vertices of exactly one child; hence, there are $\binom{n+k-1}{j}$ ways to put these j vertices. Therefore,

$$\begin{aligned} |\mathbb{F}_{n,k}(1,\vec{s})| &= \sum_{j\geq 0} \binom{n+k-1}{j} |\mathbb{F}_{n-j,k}(\vec{s})| \\ &= \sum_{i\geq 0} \binom{n+k-1}{n-i} |\mathbb{F}_{i,k}(\vec{s})|. \end{aligned}$$

2. With the same argument as above, if we add $n-i(\geq 0)$ vertices to the plane forests in $\mathbb{F}_{i,k}(1,\vec{s})$ such that each new forest has at least $n-i$ vertices with one child, then there are $\binom{n+k-1}{n-i}|\mathbb{F}_{i,k}(1,\vec{s})|$ possibilities. Each forest in this construction repeats $\binom{r}{n-i}$ times if it contains exactly $r(\geq n-i)$ vertices with one child. Let $f \in \mathbb{F}_{n,k}(1,\vec{s})$ has r vertices with one child. Then f appears $\binom{r}{n-i}$ times among $\binom{n+k-1}{n-i}|\mathbb{F}_{i,k}(1,\vec{s})|$ plane forests. Hence, the number of f counted at the right side of the identity is $\binom{r}{0}-\binom{r}{1}+\binom{r}{2}\cdots+(-1)^r\binom{r}{r}$ which equals to the number of f counted at the left side of the identity and the proof follows.

❑

Example 4.3.2 and Example 4.3.3 are applications of Theorem 4.3.1-1 and Theorem 4.3.1-2, respectively.

Example 4.3.2 *It is well-known that* $M_{n,k} = |\mathbb{F}_{n,k}(1,2)|$ *and* $C_{n,k} = |\mathbb{F}_{2n,k}(2)|$. *Hence we immediately obtain the relation of* $M_{n,k}$ *and* $C_{i,k}$, *i.e.*

$$\begin{aligned} M_{n,k} &= |\mathbb{F}_{n,k}(1,2)| \\ &= \sum_{j\geq 0}\binom{n+k-1}{n-j}|\mathbb{F}_{j,k}(2)|\ ,\ \textit{by Theorem 4.3.1-1,} \\ &= \sum_{i\geq 0}\binom{n+k-1}{n-2i}|\mathbb{F}_{2i,k}(2)| \\ &= \sum_{i\geq 0}\binom{n+k-1}{n-2i}C_{i,k}. \end{aligned}$$

In particular, as $k=1$, *we have* $M_n = \sum_{i\geq 0}\binom{n}{2i}C_i$.

Similarly,

$$\begin{aligned} C_{n,k} &= |\mathbb{F}_{n,k}(1,\vec{s})|, \textit{where } \vec{s}=(2,3,...,n), \\ &= \sum_{i\geq 0}\binom{n+k-1}{n-i}|\mathbb{F}_{i,k}(\vec{s})|\ ,\ \textit{by Theorem 4.3.1-1,} \\ &= \sum_{i\geq 0}\binom{n+k-1}{n-i}R_{i,k}. \end{aligned}$$

In particular, as $k=1$, *we have* $C_n = \sum_{i\geq 0}\binom{n}{i}R_i$.

Example 4.3.3 *With an analogous argument as above, we immediately obtain the relation between $C_{n,k}$ and $M_{i,k}$, i.e.*

$$\begin{aligned} C_{n,k} &= |\mathbb{F}_{2n,k}(2)| \\ &= \sum_{i\geq 0}(-1)^{2n-i}\binom{2n+k-1}{2n-i}|\mathbb{F}_{i,k}(1,2)| \text{ ,by Theorem 4.3.1-2,} \\ &= \sum_{i\geq 0}(-1)^{2n-i}\binom{2n+k-1}{2n-i}M_{i,k}. \end{aligned}$$

In particular, as $k=1$, we have $C_n = \sum_{i\geq 0}(-1)^{2n-i}\binom{2n}{i}M_i$.

Similarly,

$$\begin{aligned} R_{n,k} &= |\mathbb{F}_{n,k}(\vec{s})|, \text{ where } \vec{s} = (2,3,...,n), \\ &= \sum_{i\geq 0}(-1)^{n-i}\binom{n+k-1}{n-i}|\mathbb{F}_{i,k}(1,\vec{s})| \text{ , by Theorem 4.3.1-2,} \\ &= \sum_{i\geq 0}(-1)^{n-i}\binom{n+k-1}{n-i}C_{i,k}. \end{aligned}$$

In particular, as $k=1$, we have $R_n = \sum_{i\geq 0}(-1)^{n-i}\binom{n}{i}C_i$.

Eu-Liu-Yeh [32] show that the Motzkin number M_{n-1} counts $n-$Dyck paths without peaks over even heights in Theorem 1 and the Riordan number R_n counts $n-$Dyck paths without peaks over odd heights in Theorem 2, i.e., M_{n-1} and R_n count $n-$plane trees with odd and even lengths from root to each leaf, respectively. Therefore, we have the following results. In $\mathbb{F}_{n,k}$,

1. the number of forests with odd length from each root to each leaf equals to $M_{n-k,k}$, and
2. the number of forests with even length from the root to each leaf equals to $R_{n,k}$.

Since $\mathbb{D}^*_{n,k}$ has two to one correspondence to $\mathbb{F}^*_{n,k}$, we obtain:

1. $2M_{n-k,k}$ counts $n-$Dyck paths with flaws and k components where peaks and valleys are of odd height, and

2. $2\sum_{i\geq 0}(-1)^i\binom{n}{i}R_{n,k-i}$ counts $n-$Dyck paths with flaws and k components where peaks and valleys are of even height.

4.4 Some Riordan Families

In [7], p. 85, we learn that the Riordan number R_n counts four families: tall bushes with $n+1$ edges, short bushes with n edges, feasible non-crossing partitions, and non-crossing partitions with no singletons, respectively. In Theorem 2 ([32], p. 456), Eu-Liu-Yeh list three Riordan families: $n-$Motzkin paths without level steps on the $x-$axis, $(n-1)-$Motzkin paths with at least one level step on the $x-$axis, and $n-$Dyck paths without peaks of odd heights, respectively. Note that in the origin article, the second result $(n+1)-$Motzkin paths with at least one level step on the $x-$axis should be $(n-1)-$Motzkin paths with at least one level step on the $x-$axis.

In this section, we shall present six different Riordan families using bijection: four families of 2-ary trees and two families of Motzkin paths.

Theorem 4.4.1 *The Riordan number R_n counts the following families:*

1. *2-ary trees with n edges where no vertex in the path from the root to the rightmost leave has only one child,*
2. *$n-$Motzkin paths without level steps before the first peak,*
3. *2-ary trees with n edges where the last internal vertex has exactly 2 children in preorder,*
4. *2-ary trees with $n-1$ edges where at least one vertex in the path from the root to the rightmost leave has only one child,*
5. *$(n-1)-$Motzkin paths with at least one level steps before the first peak, and*
6. *2-ary trees with $n-1$ edges where the last internal vertex has exactly one child in preorder.*

Proof.

1. For any short bush, we label the root 0. And we label the first child, the last child, and the other with 2,0, and 1, respectively. Collect these numbers in preorder and rearrange them in preorder by letting the labels denote the number of children. This construction yield a 2-ary tree where no vertices in the path from the root to the righmost leave have only one child. Figure 4.5-(a) is an example with $n = 12$ for illustrating this bijection.

2. By symmetry, the Riordan number also counts 2-ary trees where no vertex in the path from the root to the leftmost leave has only one child. If we correspond 2, 0 and 1 to rise step u, fall step d, and level step l, respectively, then we get an $n-$Motzkin path without level steps before the first peak, where $u = (1, 1)$, $d = (1, -1)$, and $l = (1, 0)$. Figure 4.5-(b) is an example with $n = 12$ for illustrating this bijection.

3. By symmetry, the Riordan number counts $n-$Motzkin paths without level steps after the last peak. As the above corresponding, $n-$Motzkin paths without level steps after the last peak correspond to 2-ary trees with n edges where the last internal vertex has exactly 2 children in preorder. Figure 4.5-(c) is an example with $n = 12$ for illustrating this bijection.

   Since the $M_n = R_{n+1} + R_n$, 4,5, and 6 follow from 1, 2, and 3, respectively.

❑

Now we return to seek another formula of $R_{n,k}$, even if it is not a closed form. On the one hand, since $R_{n,k}$ counts plane forests with n edges ($n+k$ vertices) and k components where each component is a 2-ary tree with the rightmost path having no vertex with one child. If we delete the rightmost path of each component, then the remaining is a plane forest with $n - j$ vertices and j ($\geq k$) components and vertices having degree at most 2. On the other hand, for a plane forest with $n - j$ vertices and j components where each component is a 2-ary tree, by Theorem 5.3.10 in [74], there are

$$\sum_{i\geq 0} \binom{j-1}{k-1} \frac{j}{n-j} \binom{n-j}{i, i+j, n-2j-2i}$$

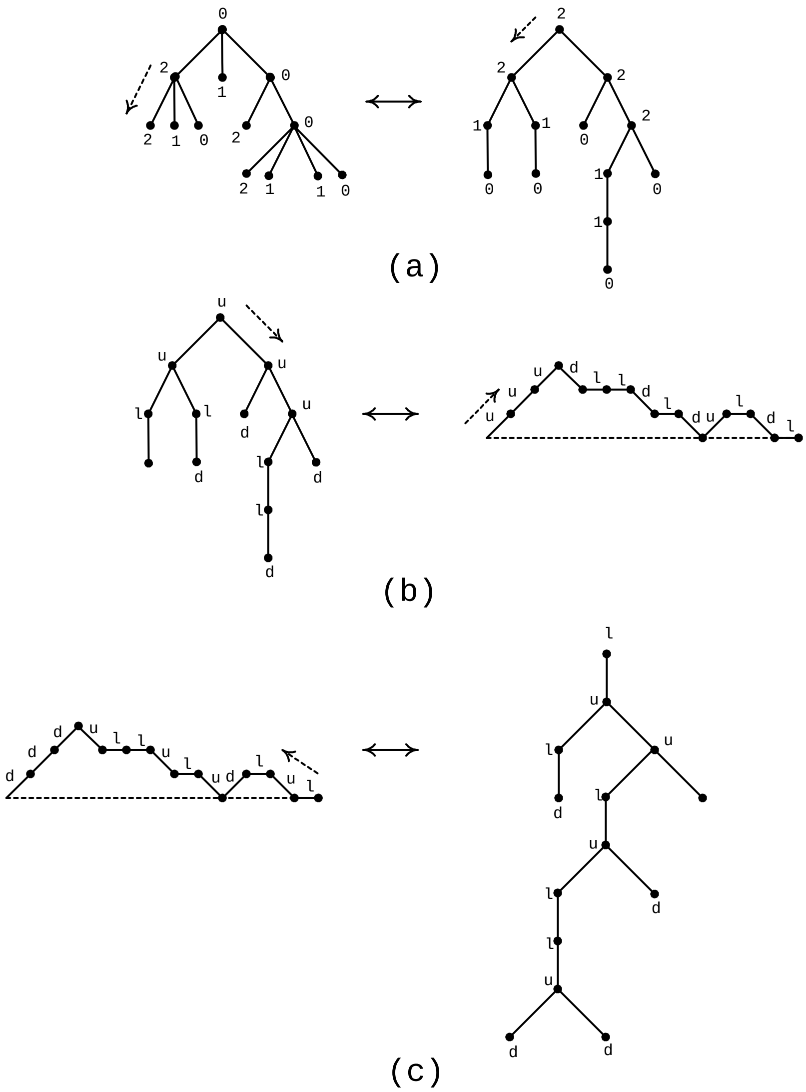

Figure 4.5: An illuatration for the proof of Theorem 4.4.1.

ways to form a plane forest with n edges ($n+k$ vertices) and k components where each component is a 2-ary tree with a rightmost path having no vertex with one child. Hence, we obtain

$$R_{n,k} = \sum_{j \geq k} \sum_{i \geq 0} \binom{j-1}{k-1} \frac{j}{n-j} \binom{n-j}{i, i+j, n-2i-2j}. \tag{4.4.1}$$

In particular,

$$R_n = \sum_{k \geq 1} \sum_{i \geq 0} \frac{k}{n-k} \binom{n-k}{i, i+k, n-2k-2i}. \tag{4.4.2}$$