Introduction

In this thesis, we shall focus on two independent topics in combinatorics: One is the constructions of Hadamard matrices and the other is the studies of forests. In each topic, we reproduce three articles either already published, to appear or submitted for publications. In Part I, Chapter 1 is our published article "On J_m−Hadamard Matrices" [69], Chapter 2 is abridged of our accepted article "On Marrero's J_m−Hadamard Matrices" [70], and Chapter 3 is our paper "On Craigen-de Launey's Constructions of Hadamard Matrices" [71] to be submitted soon.

In Part II, Chapter 4 is our article "On Enumeration of Plane Forests" [16], Chapter 5 is an extended version of the paper "The Chung-Feller Theorem Revisited" [17], and Chapter 6 extends our submitted paper "2-Caterpillars are graceful" [18].

I. Constructions of Hadamard Matrices

In this topic, our main purpose is to construct Hadamard Matrices from a given one or given ones. One is to construct other Hadamard matrices from a given J_m−Hadamard matrix (Definition 1.1.1), and the other is to construct an Hadamard matrix of order $2^k m_1 m_2 \cdots m_t$ from given t Hadamard matrices of order $4m_1, 4m_2, ...,$ $4m_t$, respectively, such that k is as small as possible.

A recent construction by Marrero [46] allows us to yield three other Hadamard matrices from a given one as follows: Let H be any $2t \times 2t$ Hadamard matrix. Then H can be transformed into the following form:

$$H \sim \begin{pmatrix} J & J & A \\ J & -J & B \end{pmatrix} = \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes J \,\middle|\, \begin{matrix} A \\ B \end{matrix} \right),$$

where $\otimes$ is the Kronecker product, $J \in \mathbb{M}_{t\times 1}(\{1\})$ and $A, B \in \mathbb{M}_{t\times(2t-2)}(\{\pm 1\})$.

Moreover, $\begin{pmatrix} J & J & -A \\ J & -J & -B \end{pmatrix}$, $\begin{pmatrix} J & J & -A \\ J & -J & B \end{pmatrix}$ and $\begin{pmatrix} J & J & A \\ J & -J & -B \end{pmatrix}$ are all Hadamard matrices.

In Chapter 1 (see [69]), we aim to generalize Marrero's result. The main results are assorted as follows:

(i) For any J_m−Hadamard matrix, we can generate $2^m - 1$ other Hadamard matrices (Theorem 1.1.2).

(ii) The Sylvester's construction ensures the existence of a J_m−Hadamard matrix provided that there is an Hadamard matrix of order m (Theorem 1.1.3 and Corollary 1.1.4).

(iii) The Kronecker product of a J_k−Hadamard matrix and an Hadamard matrices with order h yields another Hadamard matrix equivalent to a J_{hk}−Hadamard matrix (Theorem 1.1.5).

(iv) Not all Hadamard matrices are J_4−Hadamard matrices (Example 1.2.1 and Example 1.2.2).

In Chapter 2 (see [70]), we revisit and strengthen results in Chapter 1 and introduce the concept of J_m−classes CJ_m.

(i) For any J_m−Hadamard matrix, we can generate $2^m m! - 1$ other Hadamard matrices by allowing permutations on S_m (Theorem 2.1.1).

(ii) From an Hadamard matrix of order $4h$ and a J_{4k}−Hadmard matrix, there exists a J_{8hk}−Hadamard matrix (Theorem 2.1.5).

(iii) All Hadamard matrices of order $12h$ and $20h$ don't belong to CJ_{4h} (Example 2.2.1 and Example 2.2.2).

(iv) $CJ_8 \subsetneqq CJ_4 \subsetneqq CJ_2$ (Theorem 2.2.3).

The Kronecker product of t Hadamard matrices is another important subject. Sylvester [77], Hadamard [37], Agayan [1], and Craigen [20] formulated the result for $t = 2$; Craigen-Seberry-Zhang [21] discussed the case for $t = 4$, and finally de Launey [22] studied the case for $t = 12$. In Chapter 3 (see [71]), we study the case for general t and the main results are associated as follows:

(i) Using de Launey's construction, we can generalize Craigen's Theorem 1 in [20] (Theorem 3.1.1).

(ii) Using our Theorem 3.1.1, we can generalize Craigen-Seberry-Zhang's Theorem 1 in [21] (Theorem 3.1.2).

(iii) We study the minimum exponent E_t of an Hadamard matrix resulting from t Hadamard matrices which is an increasing step function (Lemma 3.2.1).

(iv) By suitably partitioning into three groups from the given Hadamard matrices, we may yield an upper bound of the minimum exponent (Theorem 3.2.4 and Corollary 3.2.5).

II. Studies of Forests

In this topic, we devote ourselves to studying three subjects in forests. We first generalize some results of plane trees to plane forests. Secondly, we are interested in Chung-Feller Theorem and discuss some related results. Thirdly, we study graceful labellings of some $n-$caterpillars. In particular, Latin squares are applied to yield graceful labellings of 2^n-caterpillars.

A great many of formulas about plane trees have been investigated, e.g., the Catalan number C_n counts plane trees of n edges. For more information of Catalan families, we refer to [7, 74]. A famous Catalan identity is the Touchard's identity [79].

Shapiro [65] used generating functions to find that among the vertices of plane trees, exactly half of them are leaves. Let e_n and o_n be the numbers of plane trees of n edges with even and odd number of leaves, respectively. Eu-Liu-Yeh [33] also used generating functions to prove that $e_{2n} - o_{2n} = 0$ and $e_{2n+1} - o_{2n+1} = (-1)^{n+1}C_n$. Using bijection, Dershowitz-Zaks [23] discovered that the Narayana number $N(n, i)$ counts plane trees with n edges and i leaves.

Bernhart [7], using linear operator, presented 6 identities including four Motzkin-Catalan identities and two Catalan-Riordan identities. Eu-Liu-Yeh [32] listed three new Riordan families: $n-$Motzkin paths without level steps on the $x-$axis, $(n-1)-$Motzkin paths with at least one level step on the $x-$axis, and $n-$Dyck paths without peaks of odd heights, respectively.

Motivated by the above results, we will generalize them to plane forests in Chapter 4 (see [16]). The main results are arranged as follows:

(i) We study the number of plane forests with x_i vertices at level i and get a new Catalan identity (Theorem 4.1.3).

(ii) We generalize a Shapiro's result: Among the vertices of plane forests with n edges and k components, exactly half of them are leaves (Theorem 4.2.1).

(iii) Let $e_{n,k}$ and $o_{n,k}$ be the numbers of plane forests with even number of leaves and odd number of leaves, respectively, where each forest has n edges and k nontrivial components. Using generating function, we discover that $e_{n,k} - o_{n,k} = 0$, if $n+k$ is odd; $(-1)^{\frac{n+k}{2}} \frac{k}{n} \binom{n}{\frac{n+k}{2}}$, otherwise (Theorem 4.2.2).

(iv) Using a bijective proof, we provide a formula to count plane forests with n edges, k components and i nontrivial leaves (Theorem 4.2.4).

(v) We generalize Motzkin-Catalan identity and Catalan-Riordan identity by establishing the relation between plane forests with vertices allowing one child and plane forests without vertices having one child (Theorem 4.3.1).

(vi) We present six new Riordan families by a bijective correspondence between u, l, d labels and $up, level, down$ steps (Theorem 4.4.1).

In Chapter 5, another combinatorial theme related to plane forests is Dyck paths with flaws. The most famous result is the Chung-Feller Theorem [19]. Recently, in 2005, Eu-Fu-Yeh [31] used Taylor expansion for Catalan number to reprove Chung-Feller Theorem. The main results are classified as follows:

(i) We reprove the Chung-Feller Theorem by a simple bijection (Theorem 5.1.1) and the Catalan identity appeared in Theorem 4.1.1 (Theorem 5.1.2).

(ii) We study the relation between the Chung-Feller Theorem and bi-color plane forests (Theorem 5.2.3).

(iii) We catch two results: One is the number of semi-standard tableaux of shape $2 \times n$ with k decreasing columns is independent of k (Theorem 5.3.2), and the other is the number of noncrossing semi-ordered pairs with n pairs and k $d-$arcs only depends on n for $k = 0, 1, 2, ..., n$ (Theorem 5.3.4).

(iv) We study Chung-Feller Theorem for Motzkin number (Theorem 5.4.1) and apply it to probability theory (Corollary 5.4.3).

(v) We study Chung-Feller Theorem for Riordan number (Theorem 5.4.4).

In Chapter 6 (see [18]), we discuss graceful labellings of $n-$caterpillars. In 1967, Rosa [60] introduced the notion of graceful labelling originally called $\beta-$ valuation and was renamed as such by Golomb in [36]. It is an open problem that trees have graceful labellings. This now notorious open problem has been known variously as Rosa's conjecture, Ringel's conjecture, or the graceful tree conjecture. Kotzig [40] has called it a disease of graph theory. A good reference for graceful labelling is a survey paper by J. A. Gallian [34].

In 1979, Bermond [6] proposed a still open conjecture that lobsters are graceful. Recently, in 2005, Mishra and Pangrahi [47] proved that some classes of lobsters have graceful labelling. For other families of graceful lobsters, we refer to [15, 48, 53, 82]. The main results are assorted as follows:

(i) We make use of an algorithm by partitioning a 2-caterpillar into union of 2-stars to yield graceful labellings of 2-caterpillars (Theorem 6.1.2).

(ii) We use iterated steps to yield graceful labellings of regular $n-$caterpillars (Theorem 6.2.3).

(iii) We prove that if the single path of a $n-$caterpillar T has no leg except in the $ni + 1^{st}$ vertex, then T has a graceful labelling (Proposition 6.3.11).

(iv) Using iterated steps, we construct a symmetric graceful Latin square of order 2^n for $n \in \mathbb{N}$ (Theorem 6.4.4).

(v) Using (ii), Theorem 6.2.3 and (iv), Theorem 6.4.4, we present graceful labellings of 2^n-caterpillars (Theorem 6.4.9).

Concluding Remarks

In this part, we indicate some further directions of our research in coming years.