Chapter 2

Further Results on J_m−Hadamard Matrices

In Chapter 1, we generalized Marrero's construction of J_2−Hadamard matrices to J_m−Hadamard matrices, $m = 2$ or $m = 4k$, $k \in \mathbb{N}$. A Marrero's J_2−Hadamard matrix (see [46]) is a normalized Hadamard matrix of order $2t$ of the form

$$\begin{pmatrix} J & J & A \\ J & -J & B \end{pmatrix},$$

where $J \in \mathbb{M}_{t\times 1}(\{1\})$ and $A, B \in \mathbb{M}_{t\times(2t-2)}(\{\pm 1\})$. By changing A into $-A$ or B into $-B$, he yielded other $2^2 - 1$ Hadamard matrices from the given one. A J_m−Hadamard matrix is an Hadamard matrix of order mt of the form

$$\left(\begin{array}{c|c} & A_1 \\ & A_2 \\ M \otimes J & \vdots \\ & A_m \end{array}\right),$$

where M is an Hadamard matrix of order m, $J \in \mathbb{M}_{t\times 1}(\{1\})$,$A_1, A_2, \cdots A_m \in \mathbb{M}_{t\times(mt-m)}(\{\pm 1\})$, and $\otimes$ is the Kronecker product (see [69], Definition 2.1). By changing A_i to $\pm A_i$, we constructed other $2^m - 1$ Hadamard matrices ([69], Theorem 2.2).

In Section 2.1, by revisiting and simplifying the proof of the above result, it turns out that we can yield other $2^m m! - 1$ Hadamard matrices by allowing permutations on $\{1,2,...,m\}$ $\sigma \in S_m$ (Theorem 2.1.1 and Remark below). In fact, if we transform A_i mentioned above into $\pm A_{\sigma(i)}$ for $i = 1,2,...,m$, where σ is a permutation of the set $\{1,2,...,m\}$, then the new matrices are still J_m-Hadamard matrices (Theorem 2.1.1). Thus we can construct other $2^m m! - 1$ Hadamard matrices from a given J_m-Hadamard matrix. Moreover, for a given Hadamard matrix of order $4k$ and another $J_{4h}-$Hadamard matrix, the Kronecker product enables us to yield a $J_{16kh}-$Hadamard matrix (Theorem 2.1.4). Continuing this process, one easily gets a $J_{2^{2n+2}k_1k_2\cdots k_nh}-$ Hadamard matrix from given n Hadamard matrices of orders $4k_1, 4k_2, ..., 4k_n$, respectively, and a $J_{4h}-$Hadamard matrix. On the other hand, there is another technique due to Craigen to construct a $J_{2^lh}-$Hadamard matrix with smaller 2-exponent l from the given Hadamard matrices: In Theorem 2.1.5, we use Craigen's construction (see [20], Theorem 1) to generate a $J_{8kh}-$Hadamard matrix from a given Hadamard matrix of order $4k$ and another $J_{4h}-$Hadamard matrix.

In Section 2.2, we introduce the concept of J_m-*classes*, $m = 2$ or $m = 4k, k \in \mathbb{N}$, denoted by CJ_m which contains the equivalent class of J_m-Hadamard matrices. By Marrero's approach, each Hadamard matrix belongs to CJ_2. For a given Hadamard matrix, it seems difficult to determine to which CJ_m it belongs. Nevertheless, we can decide to which CJ_m it doesn't belong. Example 2.2.1 and Example 2.2.2 prove that an Hadamard matrix of order $12h$ or $20h$ doesn't belong to CJ_{4h}. Here the question about whether $CJ_{n'} \subseteq CJ_n$ for $n \mid n'$ is studied. Our initial contribution to this question is to show $CJ_8 \subsetneq CJ_4 \subsetneq CJ_2$ (Theorem 2.2.3).

We end this chapter by leaving the question open whether for a given n, $CJ_{2^n} \subseteq CJ_{2^m}$ for some $1 \neq m < n$.

2.1 Some Properties of J_m-Hadamard Matrices

In our previous paper [69], Theorem 2.2, for a given J_m-Hadamard matrix H as in Introduction, we show that all the matrix of the form $\hat{H} = \left(\begin{array}{c|c} & \pm A_1 \\ M \otimes J & \pm A_2 \\ & \vdots \\ & \pm A_m \end{array}\right)$ are

J_m-Hadamard matrices, generalizing Marrero's result ([46], Proposition). In the following, we will prove a stronger result where permutations are allowed:

Theorem 2.1.1 *Let H be a J_m-Hadamard matrix of the form as above. Then*

$$\hat{H} = \left(\begin{array}{c|c} & B_1 \\ & B_2 \\ M \otimes J & \vdots \\ & B_m \end{array} \right)$$

is also a J_m-Hadamard matrix, where $B_i = A_{\sigma(i)}$ or $B_i = -A_{\sigma(i)}$ for $i = 1, 2, \ldots, m$ and $\sigma \in S_m$.

Proof. Write $M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_m \end{pmatrix}$, $A_i = \begin{pmatrix} A_{i1} \\ A_{i2} \\ \vdots \\ A_{it} \end{pmatrix}$ and $B_i = \begin{pmatrix} B_{i1} \\ B_{i2} \\ \vdots \\ B_{it} \end{pmatrix}$, where M_i, A_{ik} and B_{ik} are the row vectors of M, A_i and B_i, respectively, for $i = 1, 2, ..., m$ and $k = 1, 2, ..., t$. Then

$$H = \left(\begin{array}{cc} M_1 & A_{11} \\ M_1 & A_{12} \\ \vdots & \vdots \\ M_1 & A_{1t} \\ \hline M_2 & A_{21} \\ M_2 & A_{22} \\ \vdots & \vdots \\ M_2 & A_{2t} \\ \hline \vdots & \vdots \\ \hline M_m & A_{m1} \\ M_m & A_{m2} \\ \vdots & \vdots \\ M_m & A_{mt} \end{array} \right), \text{ and } \hat{H} = \left(\begin{array}{cc} M_1 & B_{11} \\ M_1 & B_{12} \\ \vdots & \vdots \\ M_1 & B_{1t} \\ \hline M_2 & B_{21} \\ M_2 & B_{22} \\ \vdots & \vdots \\ M_2 & B_{2t} \\ \hline \vdots & \vdots \\ \hline M_m & B_{m1} \\ M_m & B_{m2} \\ \vdots & \vdots \\ M_m & B_{mt} \end{array} \right).$$

Since H is an Hadamard matrix, then for $i, j = 1, 2, ..., m$ and $k, l = 1, 2, ..., t$, we have

$$M_i M_j^T + A_{ik} A_{jl}^T = \begin{cases} mt, & \text{if } i = j \text{ and } k = l, \\ 0, & \text{otherwise.} \end{cases}$$

This implies

$$A_{ik} A_{jl}^T = \begin{cases} mt - m, & \text{if } i = j \text{ and } k = l, \\ -m, & \text{if } i = j \text{ and } k \neq l, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.1.1}$$

It suffices to prove that $M_i M_j^T + B_{ik} B_{jl}^T = \begin{cases} mt, & \text{if } i = j \text{ and } k = l, \\ 0, & \text{otherwise.} \end{cases}$

Case 1: $i = j$ and $k = l$, i.e. $\sigma(i) = \sigma(j)$. $M_i M_j^T + B_{ik} B_{jl}^T = M_i M_i^T + B_{ik} B_{ik}^T = m + B_{ik} B_{ik}^T = M_{\sigma(i)} M_{\sigma(i)}^T + A_{\sigma(i)k} A_{\sigma(i)k}^T = mt$, by (2.1.1).

Case 2: $i = j$ and $k \neq l$. $M_i M_j^T + B_{ik} B_{jl}^T = M_i M_i^T + B_{ik} B_{il}^T = M_i M_i^T + A_{\sigma(i)k} A_{\sigma(i)l}^T = m + (-m) = 0$, by (2.1.1).

Case 3: $i \neq j$, i.e. $\sigma(i) \neq \sigma(j)$. $M_i M_j^T + B_{ik} B_{jl}^T = M_i M_j^T \pm A_{\sigma(i)k} A_{\sigma(j)l}^T = 0 + 0 = 0$, by (2.1.1). This completes the proof. ❑

Remark. It seems that one gets more Hadamard matrices from the J_m−Hadamard matrix above by also permuting rows inside each B_i, $i = 1, 2, ..., m$. However, by these permutations, one actually gets equivalent ones. Furthermore, it fails to produce Hadamard matrices if one instead permutes rows from different B_is.

Upon suggestions of Professor Gerard J. Chang, we can obtain Theorem 1.1.2 and Theorem 2.1.1 as direct consequences of the following Lemma:

Lemma 2.1.2 *Suppose that M is an Hadamard matrix. Then* $\left(M \otimes J \middle| \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \right)$ *is an Hadamard matrix if and only if* $A_i A_j^T = \delta_{i,j}\ (mtI_{t\times t} - m \begin{pmatrix} 1 \end{pmatrix}_{t\times t})$ *for* $i, j = 1, 2, ..., m$.

By Theorem 2.1.1, we may produce $2^m m! - 1$ other Hadamard matrices from a given J_m-Hadamard matrix. In passing, we note the following further characterization of Hadamard matrices which will be useful in our discussion later on J_m-Hadamard matrices (Remark at the end of Section 2.2).

Corollary 2.1.3 *Let H be a J_m-Hadamard matrix of the form as above. If M is a J_l-Hadamard matrix of the form*

$$\left(\begin{array}{c|c} & C_1 \\ & C_2 \\ L\otimes J' & \vdots \\ & C_l \end{array}\right),$$

then

$$\hat{H} = \left(\left(\begin{array}{c|c} & \pm C_{\delta(1)} \\ & \pm C_{\delta(2)} \\ L\otimes J' & \vdots \\ & \pm C_{\delta(l)} \end{array}\right)\otimes J \,\middle|\, \begin{array}{c} \pm A_{\sigma(1)} \\ \pm A_{\sigma(2)} \\ \vdots \\ \pm A_{\sigma(m)} \end{array}\right)$$

is also a J_l-Hadamard matrix, where $\sigma \in S_m$ and $\delta \in S_l$. In particular, H itself is a J_l-Hadamard matrix.

Proof. Let $\hat{M} = \left(\begin{array}{c|c} & \pm C_{\delta(1)} \\ & \pm C_{\delta(2)} \\ L\otimes J' & \vdots \\ & \pm C_{\delta(l)} \end{array}\right)$. By Theorem 2.1.1, $\hat{M}$ is a J_l-Hadamard matrix of order m and trivially $\hat{H}$ is an Hadamard matrix. It remains to prove that $\hat{H}$ is evidently a J_l-Hadamard matrix.

To this end, just put $L\otimes(J'\otimes J) = L\otimes J''$, where $J' \in \mathbb{M}_{t'\times 1}(\{1\}), J \in \mathbb{M}_{t\times 1}(\{1\})$ and $J'' \in \mathbb{M}_{tt'\times 1}(\{1\})$, here $t' = \frac{m}{l}$, then clearly,

$$\hat{H} = \left(\left(\begin{array}{c|c} & \pm C_{\delta(1)}\otimes J \\ & \pm C_{\delta(2)}\otimes J \\ L\otimes J'' & \vdots \\ & \pm C_{\delta(l)}\otimes J \end{array}\right) \,\middle|\, \begin{array}{c} \pm A_{\sigma(1)} \\ \pm A_{\sigma(2)} \\ \vdots \\ \pm A_{\sigma(m)} \end{array}\right)$$

is a J_l-Hadamard matrix and the proof follows. ❑

Next, we start with the Kronecker product of an Hadamard matrix K of order $4k$, and a $J_{4h}-$Hadamard matrix $H = (M\otimes J\,|A)$ of order $4ht$. In our previous paper [69], Theorem 2.5, using combinatorial arguments, we showed that $K \otimes H$ is equivalent to a $J_{16kh}-$Hadamard matrix $(K \otimes M \otimes J \,|K \otimes A)$. In the following, using only matrix multiplications and Kronecker product (see e.g. Craigen's paper [20], p. 57), we reprove the result as follows.

Theorem 2.1.4 *Let K be an Hadamard matrix of order $4k$. If $H = (M \otimes J \,|A)$ is a $J_{4h}-$Hadamard matrix of order $4ht$, then $K \otimes H \sim (K \otimes M \otimes J \,|K \otimes A)$ and $(K \otimes M \otimes J \,|K \otimes A)$ is a $J_{16kh}-$Hadamard matrix of order $16kht$.*

Proof. Let $\tilde{H} = (K \otimes M \otimes J \,|K \otimes A)$. Then

$$\begin{aligned} \tilde{H}\tilde{H}^T &= KK^T \otimes MM^T \otimes JJ^T + KK^T \otimes AA^T \\ &= KK^T \otimes (MM^T \otimes JJ^T + KK^T \otimes AA^T) \\ &= 4kI_{4k} \otimes 4htI_{4ht} = 16khtI_{16kht}. \end{aligned}$$

Since $K \otimes M$ is an Hadamard matrix of order $16kh$, hence $\tilde{H}$ is a J_{16kh}-Hadamard matrix. ❑

With the supposedly existing Hadamard matrices K and H as in Theorem 2.1.4, using successively Sylvester's constructions, we yield a $J_{2^{l+4}kh}-$Hadamard matrix for $l \geq 0$. Now, using Craigen's technique, we shall obtain a $J_{2^{l+4}kh}-$Hadamard matrix with $l = -1$. In fact, we have the following result which is a generalization of Craigen's Theorem 1 in [20].

Theorem 2.1.5 *If there exists a $J_{4h}-$Hadamard matrix H of order $4ht$ and an Hadamard matrix K of order $4k$, then there is a $J_{8hk}-$Hadamard matrix of order $8hkt$.*

Proof. Write $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ and $H = \left(\begin{pmatrix} H_1 & H_2 \end{pmatrix} \otimes J \,\middle|\, \begin{matrix} A_1 & A_2 \end{matrix} \right)$, where $K_i \in \mathbb{M}_{2k\times 4k}(\{\pm 1\})$, $H_i \in \mathbb{M}_{4h\times 2h}(\{\pm 1\})$, $A_i \in \mathbb{M}_{4ht\times(2ht-2h)}(\{\pm 1\})$ for $i = 1, 2$, and $J \in \mathbb{M}_{t\times 1}(\{1\})$. Since K and H both are Hadamard matrices, we have

$$K_1K_1^T = K_2K_2^T = 4kI_{2k}, \; K_1K_2^T = K_2K_1^T = O_{2k},$$

$$(H_1H_1^T + H_2H_2^T) \otimes JJ^T + A_1A_1^T + A_2A_2^T = 4htI_{4ht}.$$

As in Craigen's constructions, put

$$S = \frac{1}{2}(K_1 + K_2) \otimes H_1 + \frac{1}{2}(K_1 - K_2) \otimes H_2,$$

$$P = \frac{1}{2}(K_1 + K_2) \otimes A_1 + \frac{1}{2}(K_1 - K_2) \otimes A_2.$$

Let $\hat{H} = \left(\begin{array}{c|c} S \otimes J & P \end{array} \right)$. Since by direct calculations, $\hat{H}\hat{H}^T = SS^T \otimes JJ^T + PP^T = 8khtI_{8kht}$, we conclude that $\hat{H}$ is a $J_{8kh}-$Hadamard matrix. ❑

2.2 Hadamard Matrices in J_m-Classes

In this section, we are interested in the problem to which J_m-Hadamard matrix does a given Hadamard matrix belong? For convenience, we define such family as follows: The family of all Hadamard matrices equivalent to some J_m-Hadamard matrix is called a J_m-*class* and denoted by CJ_m.

By Marrero's construction, each Hadamard matrix belongs to CJ_2. For a given Hadamard matrix, it seems difficult to determine to which CJ_m it belongs. Nevertheless, for some particular Hadamard matrices, we can decide to which CJ_m it doesn't belong. The following two results supply us criteria for this purpose which are generalizations of Example 3.1 and Example 3.2 in [69], respectively.

Example 2.2.1 *If H is an Hadamard matrix of order $12h$, then H doesn't belong to CJ_{4h}.*

Proof. If H were equivalent to a J_{4h}-Hadamard matrix, then

$H \sim \left(\begin{array}{c|c} M \otimes J & \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_{4h} \end{matrix} \end{array} \right)$, where M is an Hadamard matrix of order $4h$, $J \in \mathbb{M}_{3\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{3\times 8h}(\{\pm 1\})$ for $i = 1, 2, ..., 4h$. By multiplying -1 to suitable rows or columns of $\left(\begin{array}{c|c} M \otimes J & \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_{4h} \end{matrix} \end{array} \right)$, M can be normalized. Hence H must be equivalent to the J_{4h}-Hadamard matrix of the form:

$$H \sim \tilde{H} = \begin{pmatrix} \overbrace{J \quad J \quad \cdots \quad J}^{4h} & A_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \end{pmatrix} = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\begin{matrix}1 & 1 & \cdots & 1\end{matrix}}^{4h}} & \\ 1 & 1 & \cdots & 1 & A_1 \\ 1 & 1 & \cdots & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right).$$

By eventually multiplying columns of $\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{4h} \end{pmatrix}$ by -1, $\tilde{H}$ can be normalized. However, $\tilde{H}$ is not an Hadamard matrix, since there are at least $4h$ 1s at the same positions between the second row and the third row contradicting to the fact that there are exactly $\frac{12h}{4}$ 1s at the same positions in both rows except the first one (see [59], Theorem 10.9, p. 429). Thus $\tilde{H}$ is not a J_{4h}-Hadamard matrix. ❑

Example 2.2.2 *If H is an Hadamard matrix of order $20h$, then H doesn't belong to CJ_{4h}.*

Proof. Suppose that H is a normalized J_{20h}-Hadamard matrix of the form as in Example 2.2.1 with $J \in \mathbb{M}_{5\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{5\times 16h}(\{\pm 1\})$ for $i = 1, 2, ..., 4h$. We will use the same argument as above to derive a contradiction by counting the number of 1s in the second, the third, the fourth and the fifth row. As before, we know that there are exactly $10h$ 1s at each row and $\frac{20h}{4}$ 1s at the same positions between any two different rows except the first one. By arranging the 1s as forward as possible, so H, with the first five rows written down, is of the following form:

$$H = \begin{pmatrix} \overbrace{J \quad J \quad \cdots \quad J}^{4h} & A_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{matrix}}^{4h} & \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ & & & \\ & & & \end{matrix}}^{h} & \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ & & & \\ & & & \end{matrix}}^{5h} & \overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ & & & \\ & & & \\ & & & \end{matrix}}^{10h} \end{pmatrix}.$$

Looking at the $(10h+1)^{th}$ column up to the $(20h)^{th}$ column , to fill in the $10h$ 1s in the third row, we need $5h$ positions in last $10h$ columns. With the same argument, to fill in the $10h$ 1s in the fourth row, we need at least $4h$ positions in the last $10h$ columns differ from the positions already taken in the third row. Finally, in the fifth row, we need at least $3h$ positions in the last $10h$ columns differ from the positions already taken in the third and the fourth rows. This means that we need in total at least $5h + 4h + 3h = 12h$ positions to fill in the 1s in the last ten columns which is impossible. Therefore, we conclude that every Hadamard matrix of order $20h$ is not equivalent to a J_{4h}-Hadamard matrix. ❑

For $n \mid n'$, the natural question is whether $CJ_{n'} \subseteq CJ_n$. We don't know the answer even whether $CJ_{2^{k+1}} \subseteq CJ_{2^k}$. Our initial contribution to this question, using Theorem 2.1.4 and Example 2.2.1, is to show the following result; this works in the special case of Hadamard matrices of order 8 which is known to be unique up to equivalence.

Theorem 2.2.3 $CJ_8 \subsetneqq CJ_4 \subsetneqq CJ_2$.

Proof. By Marrero's construction and Example 3.1 in [69], we obtain $CJ_4 \subsetneqq CJ_2$. It remains to show that $CJ_8 \subsetneqq CJ_4$.

By the uniqueness of Hadamard matrices, every J_8-Hadamard matrix of order

8t is equivalent to the following normalized Hadamard matrix (see e.g. [72])

$$\left(\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \otimes J \;\middle|\; \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{matrix}\right)$$

$$= \left(\left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; \begin{matrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{matrix}\right) \otimes J \;\middle|\; \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{matrix}\right)$$

$$= \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} J \\ J \end{pmatrix} \;\middle|\; \begin{matrix} J & J & J & J & A_1 \\ -J & -J & -J & -J & A_2 \\ J & J & -J & -J & A_3 \\ -J & -J & J & J & A_4 \\ J & -J & J & -J & A_5 \\ -J & J & -J & J & A_6 \\ J & -J & -J & J & A_7 \\ -J & J & J & -J & A_8 \end{matrix}\right),$$

where $J \in \mathbb{M}_{t\times 1}(\{1\})$ and $A_i \in \mathbb{M}_{t\times(8t-8)}(\{\pm 1\})$ for $i = 1, 2, .., 8$. This yields $CJ_8 \subseteq CJ_4$. Next, let H be an Hadamard matrix of order 12 of the form $\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes J \,|A\right)$, where $J \in \mathbb{M}_{6\times 1}(\{1\})$ and $A \in \mathbb{M}_{12\times 10}(\{\pm 1\})$.

Set $\hat{H} = \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes J \;\middle|\; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes A\right)$.

By Theorem 2.1.4, $\hat{H} \in CJ_4$. Since $\hat{H}$ is an Hadamard matrix of order 24, by

Example 2.2.1, $\hat{H}$ doesn't belong to CJ_8, and this gives $CJ_8 \subsetneqq CJ_4$. ❑

Remark. As a consequence of our Corollary 2.1.3, a J_m-Hadamard matrix H is a J_l-Hadamard matrix for some $l \mid m$, where l depends on m and H. The question whether l depends only on m is extremely difficult. However, since $CJ_8 \subsetneqq CJ_4 \subsetneqq CJ_2$, it seems likely that $CJ_{2^n} \subseteq CJ_{2^m}$ for some $1 \neq m < n$.