Chapter 6

Some Generalization of Nevanlinna's Five-Values Theorem

6.1 Introduction

Nevanlinna's five-value theorem [25] says that if two meromorphic functions share five distinct values ignoring multiplicity, then these two functions must be identical. More precisely, suppose f and g are meromorphic functions and $a_1, a_2, \ldots, a_5$ are five distinct values. If

$$\overline{E}(a_i, f) = \overline{E}(a_i, g), \ 1 \leq i \leq 5$$

where $\overline{E}(a, h) = \{z \mid h(z) - a = 0\}$ for a meromorphic function $h(z)$, then $f \equiv g$.

C. C. Yang [35] observed that one can weaken the assumption of sharing five values to "partially" sharing five values in Nevanlinna's five-value theorem. We say that a meromorphic function f partially shares a value a with a meromorphic function g if

$$\overline{E}(a, f) \subset \overline{E}(a, g).$$

Under this terminology, Yang [35] proved that if a meromorphic function f partially

share five values $a_1, a_2, \ldots, a_5$ with a meromorphic function g, and

$$\liminf_{r \to \infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_j}\right) \Big/ \sum_{i=1}^{5} \overline{N}\left(r, \frac{1}{g - a_i}\right) > \frac{1}{2},$$

then f and g must be identical. In Nevanlinna's five-value theorem, we have $\overline{E}(a_i, f) = \overline{E}(a_i, g)$ for all $1 \le i \le 5$. In this case,

$$\liminf_{r \to \infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_i}\right) \Big/ \sum_{i=1}^{5} \overline{N}\left(r, \frac{1}{g - a_i}\right) = 1 > \frac{1}{2},$$

so $f \equiv g$. Hence Yang's result is a generalization of Nevanlinna's five-value theorem.

In this chapter, we generalize Yang's result to two meromorphic functions partially share either five or more values, or five or more small functions, and get the following main results.

Theorem A Let f and g be two non-constant meromorphic functions and a_1, a_2, $\ldots$, a_k, be k distinct values, where $k \ge 5$ and $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g)$ for all $1 \le i \le k$. If $f \not\equiv g$, then

$$\liminf_{r \to \infty} \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f - a_i}\right) \Big/ \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g - a_i}\right) \le \frac{1}{k - 3}.$$

When $k = 5$, our theorem is exactly Yang's result, so Theorem A is a generalization to both Yang's result and Nevanlinna's five-value theorem.

Li and Qiao [17] proved a small function version of Nevanlinna's five-value theorem, which says that if two meromorphic functions share five small functions, then these two functions are identical.

Our theorem for meromorphic functions partially sharing values can also be extend to partially sharing small functions.

Theorem B Let f and g be two non-constant meromorphic functions and $a_1(z)$, $a_2(z)$, $\ldots$, $a_k(z)$, be k distinct small functions of f and g, where $k \ge 5$, and $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g)$ for all $1 \le i \le k$. If $f \not\equiv g$, then

$$\liminf_{r \to \infty} \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f - a_i}\right) \Big/ \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g - a_i}\right) \le \frac{5}{2k - 5}.$$

In Theorem A, Nevalinna's second fundamental theorem is the main tool we use in our proof. Note that in the proof of Theorem B, we cannot use the Nevalinna's second fundamental theorem, because its precise form concerning small functions still remains open [33]. Instead, we use a result of Yang [35]. That is the reason two main theorems have different formulations.

6.2 Meromorphic Functions Partially Share Values

Definition 6.2.1 *Let $h(z)$ be a non-constant meromorphic function and a be a value in the extended complex plane. We define*

$$\overline{E}(a,h) = \{z \mid h(z) - a = 0\}$$

in which each zero is counted only once.

In this section, we study two meromorphic functions partially share five or more values. Precisely speaking, we consider two meromorphic functions f and g, and k distinct values $a_1, a_2, \ldots, a_k$, $k \geq 5$, such that

$$\overline{E}(a_i, f) \subset \overline{E}(a_i, g),$$

for all $1 \leq i \leq k$.

When $k = 5$, Yang [35] proved the following theorem.

Theorem 6.2.2 *Let f and g be two non-constant meromorphic functions and a_1, a_2, ..., a_5 be five distinct values. If*

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g),$$

for all $1 \leq i \leq 5$, and

$$\liminf_{r \to \infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g - a_j}\right) > \frac{1}{2},$$

then $f \equiv g$.

In the proof of this theorem, Yang gave an argument to show that if $f \not\equiv g$, then

$$\liminf_{r\to\infty} \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) \Big/ \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) \leq \frac{1}{2}, \qquad (6.2.1)$$

and hence the theorem is true. The inequality (6.2.1) is the crucial part of this theorem. It is a natural question to ask: if f and g partially share more than five values, what the corresponding inequality becomes? In this chapter, we answer this question completely by the following theorem.

Theorem A Let f and g be two non-constant meromorphic functions and a_1, a_2, ..., a_k be k distinct values, where $k \geq 5$, and $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g)$ for all $1 \leq i \leq k$. If $f \not\equiv g$, then

$$\liminf_{r\to\infty} \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) \Big/ \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g-a_i}\right) \leq \frac{1}{k-3}.$$

Proof. Without loss of generality, we may assume that all a_i are finite. By Nevanlinna's second fundamental theorem, we have

$$(k-2)T(r, f) < \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r, f)$$

and

$$(k-2)T(r, g) < \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g-a_i}\right) + S(r, g)$$

By the hypothesis $f \not\equiv g$, and $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g)$, $1 \leq i \leq k$, we have

$$\sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) \leq \overline{N}\left(r, \frac{1}{f-g}\right) \leq T(r, f) + T(r, g) + O(1).$$

Hence,

$$\sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) \leq \left(\frac{1}{k-2} + o(1)\right) \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) +$$

$$\left(\frac{1}{k-2} + o(1)\right) \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g-a_i}\right)$$

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for $r \notin E$, which implies

$$\left(\frac{k-3}{k-2} + o(1)\right) \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) \le \left(\frac{1}{k-2} + o(1)\right) \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g-a_i}\right)$$

for $r \notin E$. Therefore, we obtain

$$\liminf_{r \to \infty} \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) \Big/ \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g-a_i}\right) \le \frac{1}{k-3},$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ❑

From Theorem A, we immediately have the following corollary, which generalizes Theorem 6.2.2, and Nevanlinna's five-value theorem.

Corollary 6.2.3 *Let f and g be two non-constant meromorphic functions and $a_1, a_2, \ldots, a_k$, be k distinct values, where $k \ge 5$, and $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g)$ for all $1 \le i \le k$. If*

$$\liminf_{r \to \infty} \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f-a_i}\right) \Big/ \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g-a_i}\right) > \frac{1}{k-3},$$

then $f \equiv g$.

6.3   Meromorphic Functions Partially Share Small Functions

We say that two non-constant meromorphic functions share a function $a(z)$ if we have $f(z) - a(z) = 0$ if and only if $g(z) - a(z) = 0$. For meromorphic functions sharing small functions, Zhang [40] proved the following theorem.

Theorem 6.3.1 *Let f and g be two non-constant meromorphic functions, and $a_1(z), a_2(z), \ldots, a_6(z)$, be six distinct small functions of f and g. If f and g share $a_1(z), a_2(z), \ldots, a_6(z)$, then $f \equiv g$.*

Li and Qiao[17] improved Theorem 6.3.1 as follows.

Theorem 6.3.2 *Let f and g be two non-constant meromorphic functions, and $a_1(z), a_2(z), \ldots, a_5(z)$, be five distinct small functions of f and g. If f and g share $a_1(z), a_2(z), \ldots, a_5(z)$, then $f \equiv g$.*

Now, we consider the case that two meromorphic functions partially share small functions.

Definition 6.3.3 *Let h be a non-constant meromorphic function and $a(z)$ be a small function of h. We define*

$$\overline{E}(a, h) = \{z | h(z) - a(z) = 0\}$$

in which each zero is counted only once.

In order to prove Theorem B, we need the following lemma [35].

Lemma 6.3.4 *Let h be a non-constant meromorphic function and $a_1(z), a_2(z), \ldots, a_5(z)$ be five distinct small functions of h. Then*

$$2T(r, h) \leq \sum_{i=1}^{5} \overline{N}(r, \frac{1}{h - a_i}) + S(r, h).$$

Theorem B Let f and g be two non-constant meromorphic functions and $a_1(z)$, $a_2(z)$, $\ldots$, $a_k(z)$, be k distinct small functions of f and g, where $k \geq 5$, and $\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g)$ for all $1 \leq i \leq k$. If $f \not\equiv g$, then

$$\liminf_{r \to \infty} \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f - a_i}\right) \Big/ \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g - a_i}\right) \leq \frac{5}{2k - 5}.$$

Proof. By Lemma 6.3.4, for any distinct $\alpha_1, \ldots, \alpha_5 \in \{1, 2, \ldots, k\}$, we have

$$2T(r, f) \leq \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f - a_{\alpha_j}}\right) + S(r, f).$$

Hence,

$$2\binom{k}{5}T(r,f) \le \frac{5}{k}\binom{k}{5}\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{f-a_i}\right) + S(r,f),$$

equivalently,

$$T(r,f) \le \frac{5}{2k}\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{f-a_i}\right) + S(r,f).$$

Similarly, we have

$$T(r,g) \le \frac{5}{2k}\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{g-a_i}\right) + S(r,g).$$

By the hypothesis $f \not\equiv g$, and $\overline{E}(a_i,f) \subseteq \overline{E}(a_i,g)$, $1 \le i \le k$, we have

$$\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{f-a_i}\right) \le \overline{N}\left(r,\frac{1}{f-g}\right) \le T(r,f) + T(r,g) + O(1).$$

Hence,

$$\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{f-a_i}\right) \le \left(\frac{5}{2k}+o(1)\right)\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{f-a_i}\right) +$$

$$\left(\frac{5}{2k}+o(1)\right)\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{g-a_i}\right)$$

as $r \notin E$, which implies

$$\left(\frac{2k-5}{2k}+o(1)\right)\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{f-a_i}\right) \le \left(\frac{5}{2k}+o(1)\right)\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{g-a_i}\right)$$

for $r \notin E$. Therefore, we obtain,

$$\liminf_{r\to\infty}\sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{f-a_i}\right) \Big/ \sum_{i=1}^{k}\overline{N}\left(r,\frac{1}{g-a_i}\right) \le \frac{5}{2k-5},$$

which completes the proof. ❑

From Theorem B, we immediately have the following corollary, which generalizes Theorem 6.3.1.

Corollary 6.3.5 *Let f and g be two non-constant meromorphic functions and $a_1(z), a_2(z), \ldots, a_k(z)$ be k distinct small functions of f and g, where $k \ge 5$, and*

$\overline{E}(a_i, f) \subseteq \overline{E}(a_i, g)$ *for all* $1 \leq i \leq k$. *If*

$$\liminf_{r \to \infty} \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{f - a_i}\right) \Big/ \sum_{i=1}^{k} \overline{N}\left(r, \frac{1}{g - a_i}\right) > \frac{5}{2k - 5},$$

then $f \equiv g$.