Chapter 8

Some Results on Meromorphic Functions Sharing Four Values DM

8.1 Introduction

The relation between uniqueness and value sharing of meromorphic functions is one major problem in value distribution theory as we mentioned in section 2.6. In this chapter, we study two meromorphic functions f and g that share four values DM.

Suppose two meromorphic functions f and g share four values. In section 2.6, we know that if either f and g share three values CM and the other one IM, or f and g share two values CM and the other two IM, then f and g must share four values CM, and the conclusion of Theorem 2.6.3 hold. The case that f and g share three values IM and the other one CM is still open. Therefore, it is natural to ask whether the conclusion of Theorem 2.6.3 holds or not if f and g share four values DM. The answer is negative, and a counterexample was given by Gundersen [8]. In order to understand that two meromorphic functions share four values DM more closely, we need some terminologies.

We say that z_0 a p-fold a-point of f if $f(z)-a$ has p-fold zero at $z=z_0$. When $f(z_0)=g(z_0)=a$, we say that z_0 is a (p,q)-fold a-point of f and g if z_0 is a p-fold a-point of f and a q-fold a-point of g. In case f and g share a and all a-points are (p,q)-fold, we call a a (p,q)-fold value of f and g.

Under this terminologies, the meromorphic functions f and g given in Gundersen's example indeed share two $(1,2)$-fold values and two $(2,1)$-fold values. However, for two meromorphic functions f and g sharing four values a_1,a_2,a_3,a_4, if each a_i-point is either a $(1,2)$-fold or $(2,1)$-fold point of f and g, then Reinders [30, 31] proved that f and g are the precise form of Gundersen's example, up to some Möbius transformation. Latter, Reinders [32] gave another pair of meromorphic functions f and g sharing four values a_1,a_2,a_3,a_4, and each a_i-point is either a $(1,3)$-fold or $(3,1)$-fold point of f and g, which are also essentially unique.

It is interesting to know that whether there is any other example of two distinct non-constant meromorphic functions f and g sharing four values a_1,a_2,a_3,a_4 and each a_i-point is either a (p,q)-fold or (q,p)-fold point of f and g, for a given pair of distinct positive integers p and q.

The main result of this chapter is to show that the above assertion holds if and only if (p,q) is either $(1,2)$ or $(1,3)$, and f and g are essentially the forms given by Gundersen and Reinders.

8.2 Key Examples and Facts

In order to show that there exist two meromorphic functions sharing four values IM but not CM, Gundersen [8] gave the following example.

$$f(z)=\frac{e^{h(z)}+b}{(e^{h(z)}-b)^2},\quad g(z)=\frac{(e^{h(z)}+b)^2}{8b^2(e^{h(z)}-b)},$$

where h is a non-constant entire function and b a non-zero complex number. It is easy to check that f and g share $0, \frac{1}{b}, \infty, -\frac{1}{8b}$ DM, where 0 and $\frac{1}{b}$ are $(1,2)$-fold values of f and g, while ∞ and $-\frac{1}{8b}$ are $(2,1)$-fold values of f and g. A somewhat surprising fact is that Reinders proved Gundersen's example is essentially unique. More precisely, Reinders [30, 31] proved the following Theorem.

Theorem 8.2.1 *Let f and g be meromorphic functions that share four distinct values a_1, a_2, a_3, a_4, such that each a_i-point is either a $(1,2)$-fold or $(2,1)$-fold point of f and g. Define*

$$\hat{f}(z) = \frac{e^{h(z)} + 1}{(e^{h(z)} - 1)^2}, \ \hat{g}(z) = \frac{(e^{h(z)} + 1)^2}{8(e^{h(z)} - 1)},$$

where $h(z)$ is a non-constant entire function. Then f and g are of the forms:

$$f = L \circ \hat{f}, \ g = L \circ \hat{g},$$

where L is a Möbius transformation.

Reinders [32] latter defined the following pair of meromorphic functions F and G, and proved the following theorem.

Theorem 8.2.2 *Let*

$$F = \frac{U'}{8\sqrt{3}} \cdot \frac{U}{U+1}, \ G = \frac{U'}{8\sqrt{3}} \cdot \frac{U+4}{(U+1)^2},$$

where U is a non-constant solution of the differential equation

$$(U')^2 = 12U(U+1)(U+4).$$

Then F and G share the values $0, 1, \infty$ and -1. Every $0, 1, \infty$ and -1 point is either a $(1,3)$-fold or $(3,1)$-fold point of F and G.

Also, from a result of Reinders [32], we can get the following theorem.

Theorem 8.2.3 *Let f and g be meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 DM, such that each a_i-point is either a $(1,3)$-fold or $(3,1)$-fold point of f and g. Let F and G be the meromorphic functions defined as in Theorem 8.2.2, then f and g are of the form*

$$f = L \circ F \circ h, \quad g = L \circ G \circ h,$$

where h is a non-constant entire function and L is a Möbius transformation.

8.3 Main Result and Proof

In order to prove the main result, we need the following fact of meromorphic functions sharing four distinct values which was proved by Gundersen [10].

Theorem 8.3.1 *Let f and g be two distinct non-constant meromorphic functions and share four distinct values a_1, a_2, a_3, a_4 IM. Then*

(i) $T(r,f) = T(r,g) + S(r,f), T(r,g) = T(r,f) + S(r,g)$;

(ii) $\sum_{i=1}^{4} \overline{N}(r, \frac{1}{f-a_i}) = 2T(r,f) + S(r,f)$.

Now, we can prove our main theorem.

Theorem 8.3.2 *Let f and g be two distinct, non-constant meromorphic functions that share four values a_1, a_2, a_3, a_4 DM. Let (p,q) be a pair of positive integers with $p < q$. If each a_i-point is either a (p,q)-fold or (q,p)-fold points of f and g, then (p,q) is either $(1,2)$ or $(1,3)$. Moreover, f and g are the forms defined in Theorem 8.2.1 and 8.2.3.*

Proof. Suppose f and g share a_1, a_2, a_3, a_4. Write $k = p + q$, it is easy to see that k must be greater than two since p, q are distinct positive integers. For $k = 3$, the only possible pair of (p,q) is $(1,2)$. Then f and g are given in the form of Theorem 8.2.1. When $k=4$, the only possible pair of (p,q) is $(1,3)$. Then f and g are given in the form of Theorem 8.2.2.

Now, consider $k \geq 5$. For each shared value a_i of f and g, any a_i-point z_0 is either a (p,q)-fold or (q,p)-fold of f and g. Therefore, it is clear that

$$(p+q)\overline{N}(r, \frac{1}{f-a_i}) \leq N(r, \frac{1}{f-a_i}) + N(r, \frac{1}{g-a_i}).$$

Since $N(r, \frac{1}{f-a_i}) \leq T(r,f) + O(1)$ and $N(r, \frac{1}{g-a_i}) \leq T(r,g) + O(1)$, $1 \leq i \leq 4$, and $T(r,g) = T(r,f) + S(r,g)$ by Theorem 8.3.1, we have, for $1 \leq i \leq 4$,

$$(p+q)\overline{N}(r, \frac{1}{f-a_i}) \leq T(r,f) + T(r,f) + S(r,f) = 2T(r,f) + S(r,f).$$

Therefore, we get

$$(p+q)\sum_{i=1}^{4}\overline{N}(r, \frac{1}{f-a_i}) \leq 8T(r,f) + S(r,f).$$

Apply Theorem 8.3.1 again, which says $\sum_{i=1}^{4}\overline{N}(r, \frac{1}{f-a_i}) = 2T(r,f) + S(r,f)$, we obtain

$$2(p+q)T(r,f) \leq 8T(r,f) + S(r,f),$$

which implies that $k = p+q \leq 4$. This contradicts to our assumption that $k \geq 5$. Therefore, k cannot be greater than 4 and the theorem is proved. ❑