Chapter 1

Introduction

In 1929, R. Nevanlinna [25] proved several remarkable results in the theory of value distributions, namely, two non-constant meromorphic functions share five distinct values must be identical; if two non-constant meromorphic functions share four values CM , then one of them must be a Möbius transformation of the other. Thereafter, the field of value distribution received more interesting, leading to the growing research of the theory for decades.

Nevanlinna's results say that five values determine a meromorphic function uniquely, and if two meromorphic functions f and g share four values CM, then they are closely related. Now, a natural question arises, namely, what happen if f and g share four values IM? In 1979 and 1983, G. G. Gundersen [8, 10] proved that if two non-constant meromorphic functions f and g share either three values CM and the other one IM, or two values CM and other two IM, then f and g share four values CM, in particular, f is a Möbius transformation of g. The remaining case, namely, f and g share one values CM and the other three IM, is still open.

In view of Nevanlinna and Gundersen's results, it is interesting to know what happen if two meromorphic functions share the number of values less than four. Obviously, we can not expect that two meromorphic functions sharing three values have some relations. Therefore, in order to study the uniqueness problem of meromorphic functions, various conditions, for example, order and deficient, should be put on the functions, most results concerning this aspect can be found in [35].

In this thesis, we study the problems which are closely related to the above.

The thesis contains eight chapters. In chapter 1, we give an introduction. In chapter 2, we review the basic theory of value distribution, especially, the Nevalinna's first and second fundamental theorem.

In chapter 3, we give a unicity condition on arbitrary q meromrophic functions of class $\mathcal{A}$ which generalize a result of Jank and Terglane on a unicity result on three meromorphic functions of class $\mathcal{A}$. Also, we prove that the condition is sharp in the cases $q = 3$ and 4. Moreover, we provide a conjecture concerning this aspect.

In chapter 4, we prove the following conjecture proposed by C. C. Yang: Let $p(z)$ and $q(z)$ be two non-constant polynomials of the same degree. If there are two distinct finite complex numbers α and β satisfying $(p(z) - \alpha)(p(z) - \beta) = 0 \Leftrightarrow (q(z) - \alpha)(q(z) - \beta) = 0$, then $p(z) \equiv q(z)$ or $p(z) + q(z) \equiv \alpha + \beta$.

In chapter 5, we study the deficient values and the Nevanlinna deficiencies of meromorphic function f of the form $f(z) = P(g(z))$, where $P(z)$ is a polynomial and g is a meromorphic function with two Picard exceptional values. In fact, we prove that all deficient values of f are of the form $P(a)$, where a is a Picard exceptional value of g. Furthermore, if g is of finite order, we compute explicitly the Nevanlinna deficiencies for f. We also give an upper bound estimation for the deficiencies of f while g is of infinite order.

In chapter 6, we generalize Nevanlinna's five-value theorem to the cases that two meromorphic functions partially share either five or more values, or five or more small functions. In each case, we formulate a way to measure how far these two meromorphic functions are from sharing either values or small functions, and use this measurement to prove a uniqueness theorem.

In chapter 7, we prove some uniqueness theorems on entire functions that share a pair of values $(a, -a)$ with their derivatives, which are reformulations of some important results about uniqueness of entire functions that share values with their derivatives.

In chapter 8, we prove that if two distinct non-constant meromorphic functions f and g share four distinct values a_1, a_2, a_3, a_4 DM such that each a_i-point is either a (p, q)-fold or (q, p)-fold point of f and g, then (p, q) is either $(1, 2)$ or $(1, 3)$ and f, g are in some particular forms.