Chapter 2

Basic Theory of Value Distribution

In this chapter, we introduce and review some basic facts and notations in complex analysis and value distribution which will be used throughout the rest of the thesis. For the sake of brevity, proofs are omitted because they are standard and may be found in [4, 7, 12, 35, 38].

2.1 Poisson-Jensen's Formula

In Nevanlinna's value distribution theory, the following Poisson-Jensen's formula plays a very important role.

Theorem 2.1.1 (Poisson-Jensen's formula) *Let* $0 < R < \infty$ *and* f *be meromorphic in* $|z| \leq R$ *and* a_μ *and* b_ν *be the zeros and poles of* f *in* $|z| \leq R$, $1 \leq \mu \leq M$, $1 \leq \nu \leq N$, *respectively. If* $z = re^{i\theta}$, $0 \leq r < R$, *and* $f(z) \neq 0, \infty$, *then we have*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} d\varphi$$

$$+ \sum_{\mu=1}^{M} \log \left| \frac{R(z - a_\mu)}{R^2 - \overline{a}_\mu z} \right| - \sum_{\nu=1}^{N} \log \left| \frac{R(z - b_\nu)}{R^2 - \overline{b}_\nu z} \right|.$$

By taking $z = 0$ in Theorem 2.1.1, we get the Jensen's formula.

Theorem 2.1.2 (Jensen's formula) *Under the assumption of Theorem 2.1.1, if $f(0) \neq 0, \infty$, then we have*

$$\log|\, f(0)\,| = \frac{1}{2\pi}\int_0^{2\pi} \log\big|\, f(Re^{i\varphi})\,\big|\, d\varphi - \sum_{\mu=1}^{M} \log\frac{R}{|\, a_\mu\,|} + \sum_{\nu=1}^{N} \log\frac{R}{|\, b_\nu\,|}.$$

The assumption $f(0) \neq 0, \infty$ in Theorem 2.1.1 can be eliminated. In fact, for $0 \leq r < \infty$, let $n(r, f)$ denote the number of poles of f in $|z| \leq r$ counting multiplicities. Consider the Laurent expansion of f at the origin

$$f(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots.$$

Note that $\lambda = n(0, \frac{1}{f}) - n(0, f)$. Consider the function

$$g(z) = \begin{cases} f(z)(\frac{R}{z})^\lambda & \text{if } z \neq 0 \\ c_\lambda R^\lambda & \text{if } z = 0, \end{cases}$$

then we have the generalized Jensen's formula.

Theorem 2.1.3 (generalized Jensen's formula) *Under the assumption of Theorem 2.1.1 without the condition $f(0) \neq 0, \infty$, then we have*

$$\log|\, c_\lambda\,| = \frac{1}{2\pi}\int_0^{2\pi} \log\big|\, f(Re^{i\varphi})\,\big|\, d\varphi - \sum_{\mu=1}^{M} \log\frac{R}{|\, a_\mu\,|} - n(0, \frac{1}{f}) \log R$$

$$+ \sum_{\nu=1}^{N} \log\frac{R}{|\, b_\nu\,|} + n(0, f) \log R,$$

where c_λ is the first non-zero coefficient of the Laurent expansion of f at 0.

2.2 The Nevanlinna's First Fundamental Theorem

From now on, meromorphic function means meromorphic in the whole complex plane. First of all, we introduce the positive logarithmic function.

Definition 2.2.1 *For* $x \geq 0$,

$$\log^+ x = \max\{\log x, 0\} = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1. \end{cases}$$

Obviously, $\log^+ x$ is a continuous non-negative increasing function on $[0, \infty)$ satisfying $\log x = \log^+ x - \log^+ \dfrac{1}{x}$ and $|\log x| = \log^+ x + \log^+ \dfrac{1}{x}$.

Let f be a meromorphic function, Nevanlinna [25] introduced the following notations.

Definition 2.2.2 *For* $0 < r < \infty$,

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

Definition 2.2.3 *For* $0 < r < \infty$,

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ *denotes the number of poles of* f *in the disc* $|\, z \,| \leq t$ *counting multiplicities.* $N(r, f)$ *is called the counting function of* f.

Definition 2.2.4 *For* $0 < r < \infty$, *the function* $T(r, f)$ *defined by*

$$T(r, f) = m(r, f) + N(r, f)$$

is called the (*Nevanlinna*) *characteristic function of* f.

It is clear that $T(r,f)$ is a non-negative increasing function and a convex function of $\log r$. Let f be given in Theorem 2.1.1. It follows form the integration by parts in Riemann-Stieltjes integral, we have

$$\sum_{\mu=1}^{M}\log\frac{R}{|\,a_\mu\,|}=\int_0^R\frac{n(t,\frac{1}{f})-n(0,\frac{1}{f})}{t}dt$$

and

$$\sum_{\nu=1}^{N}\log\frac{R}{|\,b_\nu\,|}=\int_0^R\frac{n(t,f)-n(0,f)}{t}dt.$$

On the other hand, the generalized Jensen's formula can be rewritten as

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\log^+|\,f(Re^{i\varphi})\,|\,d\varphi+\sum_{\nu=1}^{N}\log\frac{R}{|\,b_\nu\,|}+n(0,f)\log R\\ =&\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f(Re^{i\varphi})}\right|d\varphi+\sum_{\mu=1}^{M}\log\frac{R}{|\,a_\mu\,|}+n(0,\frac{1}{f})\log R+\log|\,c_\lambda\,|\,.\end{aligned}$$

Therefore, we obtain

$$m(R,f)+N(R,f)=m(R,\frac{1}{f})+N(R,\frac{1}{f})+\log|\,c_\lambda\,|\,,$$

that is

$$T(R,f)=T(R,\frac{1}{f})+\log|\,c_\lambda\,|\,,$$

which is another form of the generalized Jensen's formula and is also known as the Nevanlinna-Jensen's formula.

Theorem 2.2.5 (Nevanlinna-Jensen's formula) *Let f be a meromorphic function , then, for $r>0$,*

$$T(r,f)=T(r,\frac{1}{f})+\log|\,c_\lambda\,|\,,$$

where c_λ is the first non-zero coefficient of the Laurent expansion of f at 0.

By the Nevanlinna-Jensen's formula, we can get the Nevanlinna's first fundamental theorem.

Theorem 2.2.6 (The Nevanlinna's First Fundamental Theorem) *Let f be a meromorphic function and a be a finite complex number. Then, for $r > 0$, we have*

$$T(r, \frac{1}{f-a}) = T(r, f) + \log |c_\lambda| + \varepsilon(a, r),$$

where c_λ is the first non-zero coefficient of the Laurent expansion of $\frac{1}{f-a}$ at 0, and

$$|\varepsilon(a, r)| \leq \log^+ |a| + \log 2.$$

Usually, the Nevanlinna's first fundamental theorem is written as

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1).$$

2.3 The Nevanlinna's Second Fundamental Theorem

Now, we come to the most important theorem in the theory of value distribution, namely, the Nevanlinna's second fundamental theorem.

Theorem 2.3.1 (The Nevanlinna's Second Fundamental Theorem) *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}$, $1 \leq j \leq q$, be q distinct finite values $(q \geq 2)$. Then*

$$m(r, f) + \sum_{j=1}^{q} m(r, \frac{1}{f-a_j}) \leq 2T(r, f) - N_1(r) + S(r, f),$$

where $N_1(r) = 2N(r, f) - N(r, f') + N(r, \frac{1}{f'})$ and

$$S(r, f) = m(r, \frac{f'}{f}) + m(r, \sum_{j=1}^{q} \frac{f'}{f-a_j}) + O(1).$$

Given $a \in \mathbb{C}$, by the Nevanlinna's first fundamental theorem,

$$m(r, \frac{1}{f-a}) = T(r, f) - N(r, \frac{1}{f-a}) + O(1).$$

Hence, the Nevanlinna's second fundamental theorem can be rewritten as follows.

Theorem 2.3.2 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty$, $1 \le j \le q$, be q distinct values ($q \ge 3$). Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} N(r, \frac{1}{f-a_j}) - N_1(r) + S(r,f),$$

where $N_1(r)$ and $S(r,f)$ are given as in Theorem 2.3.1.

Note that, in Theorem 2.3.2, if some $a_j = \infty$, then $N(r, \frac{1}{f-a_j})$ should be read as $N(r,f)$.

Let $n_1(t) = 2n(t,f) - n(t,f') + n(t, \frac{1}{f'})$ and let $\overline{n}(t,f)$ denote the number of distinct poles of f in $|z| \le t$. Define

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t} dt + \overline{n}(0,f) \log r,$$

which is called the reduced counting function of f. Note that, if z_0 is a pole of f of order k in $|z| \le t$, then z_0 is counted $k-1$ times by $n_1(r)$. Similarly, for a finite value a, if z_0 is a zero $f-a$ of order k in $|z| \le t$, then z_0 is also counted $k-1$ times by $n_1(r)$. Hence,

$$\sum_{j=1}^{q} N(r, \frac{1}{f-a_j}) - N_1(r) \le \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}).$$

Therefore, we have the third form of the Nevanlinna's second fundamental theorem.

Theorem 2.3.3 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty$, $1 \le j \le q$, be q distinct values ($q \ge 3$). Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

where $S(r,f)$ is given as in Theorem 2.3.1.

2.4 The Estimation of $S(r,f)$

In the Nevanlinna's second fundamental theorem, the remainder term $S(r,f)$ is a complicated object which can be estimated by using the method of logarithmic derivative. It turns out that $S(r,f)$ is small comparing to $T(r,f)$. In order to make it clear, we need the concept of the growth of meromorphic function.

Classically, we use the maximum modulus to measure the growth of an entire function.

Definition 2.4.1 *Let f be an entire function and $M(r,f) = \max_{|z|=r} |f(z)|$, $0 \leq r < \infty$. The order λ of f is defined to be*

$$\lambda = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r}$$

and the lower order is defined to be

$$\mu = \liminf_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r}.$$

For example, e^z is of order 1 and all polynomials are of order 0. However, the definition can not be applied to meromorphic functions which are not entire. Instead, we use $T(r,f)$ to measure the growth of meromorphic functions.

Definition 2.4.2 *Let f be a meromorphic function. The order λ of f is defined to be*

$$\lambda = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$$

and the lower order μ of f is defined to be

$$\mu = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Theorem 2.4.3 *Let $0 \leq r < R < \infty$ and f be an entire function, we have*

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r} T(R,f).$$

In particular,

$$T(r,f) \leq \log^+ M(r,f) \leq 3T(2r,f).$$

By Theorem 2.4.3, the order and lower order of an entire function are unambiguous. Now, we can state the properties of $S(r, f)$.

Lemma 2.4.4 *Let f be a non-constant meromorphic function. If f is of finite order, then*

$$m(r, \frac{f'}{f}) = O(\log r), \quad (r \to \infty).$$

If f is of infinite order, then

$$m(r, \frac{f'}{f}) = O(\log(rT(r, f))), \quad (r \to \infty, r \notin E),$$

where E is a set of finite measure.

Theorem 2.4.5 *Let f be a non-constant meromorphic function and $S(r, f)$ be defined in Theorem 2.3.1. If f is of finite order, then*

$$S(r, f) = O(\log r), \quad (r \to \infty).$$

If f is of infinite order, then

$$S(r, f) = O(\log(rT(r, f))), \quad (r \to \infty, r \notin E),$$

where E is a set of finite measure.

In the thesis, we will denote by $S(r, f)$ any quantity satisfy $S(r, f) = o(T(r, f))$ as $r \to \infty$ if f is of finite order, and $S(r, f) = o(T(r, f))$ as $r \to \infty, r \notin E$ if f is of infinite order, where E is a set of finite measure.

Definition 2.4.6 *Let f be a meromorphic function. A meromorphic function $a(z)$ is said to be a small function of f if $T(r, a) = S(r, f)$.*

By Lemma 2.4.4, $m(r, \frac{f'}{f}) = S(r, f)$. Moreover, Milloux [21] proved the following.

Theorem 2.4.7 *Let f be a non-constant meromorphic function and k be a positive integer and let*

$$\Psi(z) = \sum_{i=1}^{k} a_i(z) f^{(i)}(z),$$

where $a_1(z), a_2(z), \ldots, a_k(z)$ are small functions of f. Then

$$m(r, \frac{\Psi}{f}) = S(r, f).$$

2.5 Deficient Value of Meromorphic Functions

In 1929, Nevanlinna [25] introduce the quantity $\delta(a, f)$ to measure the degree of a meromorphic function misses a value a. Denote $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, the extended complex number system.

Definition 2.5.1 *Let f be a non-constant meromorphic function and $a \in \mathbb{C}_\infty$. The deficiency of a with respect to f is defined to be*

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

If $\delta(a, f) > 0$, then a is called a deficient value of f.

Definition 2.5.2 *Let f be a non-constant meromorphic function and $a \in \mathbb{C}_\infty$. We define*

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)},$$

and

$$\theta(a, f) = \liminf_{r\to\infty} \frac{N(r, \frac{1}{f-a}) - \overline{N}(r, \frac{1}{f-a})}{T(r, f)}.$$

Clearly, $0 \le \delta(a, f) \le 1$, $0 \le \Theta(a, f) \le 1$ and $0 \le \theta(a, f) \le 1$. Also, $0 \le \delta(a, f) + \theta(a, f) \le \Theta(a, f)$. By Theorem 2.3.3, we have

Theorem 2.5.3 *Let f be a non-constant meromorphic function. Then*

$$\sum_a \delta(a,f) + \theta(a,f) \leq \sum_a \Theta(a,f) \leq 2.$$

Corollary 2.5.4 *Let f be a non-constant meromorphic function. Then there are at most countably many deficient values of f and*

$$\sum_a \delta(a,f) \leq 2.$$

2.6 Some Well-Known Results on Four Value Problem

In this section, we record some well-known results on four value problem. First, we need some definitions.

Definition 2.6.1 *Let f and g be non-constant meromorphic functions and $a \in \mathbb{C}_\infty$. We say that*

(i) *f and g share a* CM *(counting multiplicities) if $f(z) - a = 0$ and $g(z) - a = 0$ have the same number of zeros with the same multiplicities.*

(ii) *f and g share a* IM *(ignoring multiplicities) if $f(z) - a = 0$ and $g(z) - a = 0$ have the same number of zeros ignoring multiplicities.*

(iii) *f and g share a* DM *(different multiplicities) if f and g share a* IM *and the zeros of $f(z) - a = 0$ and $g(z) - a = 0$ have different multiplicities at every point.*

For example, e^z and e^{-z} share $0, 1, -1, \infty$ CM; $p(z) = z^2(z-1)$ and $q(z) = z^3(z-1)^2$ share 0 DM.

In 1929, R. Nevanlinna [25] proved the following remarkable results.

Theorem 2.6.2 *If f and g are two meromorphic functions and share five distinct values in $\mathbb{C}_\infty$, then $f \equiv g$.*

Theorem 2.6.3 *If f and g are two meromorphic functions and share four distinct values a_1, a_2, a_3 and a_4 CM, then f is a Möbius transformation of g, two of the values, say a_1 and a_2, must be lacunary, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1979 and 1983, G. G. Gundersen [8, 10] proved the following results.

Theorem 2.6.4 *If f and g are two meromorphic functions and share three values* CM *and share a fourth value* IM*, then they share all four values* CM *and, hence, Theorem 2.6.3 holds.*

Theorem 2.6.5 *Let f and g be two meromorphic functions sharing four values a_1, a_2, a_3 and a_4. If f and g share a_1, a_2* CM*, and a_3, a_4* IM*, then f and g share all four values* CM *and, hence, Theorem 2.6.3 holds.*

What happens for the remaining case?, i.e., if f and g share one value CM and three values IM, can one get the same conclusions as in Theorem 2.6.3? This problem is still open and there are some partial results by putting some additional assumptions [18, 34, 35].