Chapter 3

Unicity of Meromorphic Functions of Class $\mathcal{A}$

3.1 Introduction

A meromorphic function f is of class $\mathcal{A}$ if it satisfies

$$\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) = S(r, f).$$

It includes all meromorphic functions f satisfy either $\delta(0, f) = \delta(\infty, f) = 1$ or $\Theta(0, f) = \Theta(\infty, f) = 1$. In this chapter, we study the unicity condition of q distinct meromorphic functions of class $\mathcal{A}$. Let $f_1, f_2, \ldots, f_q$ be q non-constant meromorphic functions and a be a complex number. Define $\overline{N}_0(r, a, f_1, f_2, \ldots, f_q)$ to be the reduced counting function of the common zeros of $f_j(z) - a$, $1 \leq j \leq q$, and we will simply use the notation $\overline{N}_0(r, a)$ if it is clear what functions we are referring to. We denote by E the set of r in $(0, \infty)$ with finite linear measure which may be variant in different place and denote by $S(r, f)$ any quantity which is $o(T(r, f))$ as $r \to \infty$, $r \notin E$.

Given meromorphic functions $f_1, f_2, \ldots, f_q$ of class $\mathcal{A}$. Define the number τ as follows.

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)}.$$

The main goal of this chapter is to study necessary conditions for τ to ensure that $f_1, f_2, \ldots, f_q$ are distinct. Brosch [2] proved the following result.

Theorem 3.1.1 *Let f, $g \in \mathcal{A}$, and*

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1,f,g)}{T(r,f)+T(r,g)} > \frac{1}{3}.$$

Then either $f \equiv g$ or $f \cdot g \equiv 1$.

By the theorem, we know that if f, g are distinct meromorphic functions of class $\mathcal{A}$ and $f \cdot g \not\equiv 1$, then we must have

$$\tau \leq \frac{1}{3}. \tag{3.1.1}$$

In the case of three meromorphic functions of class $\mathcal{A}$, Jank and Terglane [14] proved the following theorem.

Theorem 3.1.2 *Let f, g, $h \in \mathcal{A}$ be three distinct meromorphic functions. Then*

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1,f,g,h)}{T(r,f)+T(r,g)+T(r,h)} \leq \frac{1}{4}.$$

Also, Jank and Terglance [14] gave an example to show that the result in Theorem 3.1.2 is sharp.

To generalize the discussion above, one can ask, given q meromorphic functions, what is the necessary condition for these meromorphic functions being distinct. Observe from the above theorems, for two meromorphic functions we have $\tau \leq \frac{1}{3}$, and $\tau \leq \frac{1}{4}$ for three meromorphic functions. It is reasonable to conjecture that if $f_j \in \mathcal{A}$, $1 \leq j \leq q$, are distinct, then $\tau \leq \frac{1}{q+1}$. In fact, we will get even better conclusion as in our main theorem.

Theorem 3.1.3 *Let $f_1, f_2, \ldots, f_q$ be q distinct meromorphic functions of class $\mathcal{A}$, where $q \geq 3$. Then*

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{2}{3q}$$

when q is even, and

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{2}{3q-1}$$

when q is odd.

3.2 Some Facts About Meromorphic Functions of Class $\mathcal{A}$

In order to prove Theorem 3.1.3, we need some basic properties of meromorphic function whose proof can be found in [35].

Lemma 3.2.1 *Let $f \in \mathcal{A}$ and $k \in \mathbb{N}$. Then*

(i) $T(r, \frac{f^{(k)}}{f}) = S(r,f)$.

(ii) $T(r, f^{(k)}) = T(r,f) + S(r,f)$.

(iii) $f^{(k)} \in \mathcal{A}$.

Lemma 3.2.2 *Let $f \in \mathcal{A}$ and a be a finite non-zero number. Then*

$$\overline{N}_{1)}(r, \frac{1}{f-a}) = T(r,f) + S(r,f),$$

where $\overline{N}_{1)}(r, \frac{1}{f-a})$ denotes the reduced counting function of simple zeros of $f-a$.

Lemma 3.2.3 *Let $f, g \in \mathcal{A}$ be distinct and $\Delta = (\frac{f''}{f'} - \frac{2f'}{f-1}) - (\frac{g''}{g'} - \frac{2g'}{g-1})$. If $\Delta \equiv 0$, then $f \cdot g \equiv 1$.*

3.3 Main Results and Proofs

Now, we can prove our main theorem.

Theorem 3.1.3 *Let $f_1, f_2, \ldots, f_q$ be q distinct meromorphic functions of class $\mathcal{A}$, where $q \geq 3$. Then*

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{2}{3q}$$

when q is even, and

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{2}{3q-1}$$

when q is odd.

Proof. Set

$$\Delta_{ij} = \left(\frac{f_i''}{f_i'} - \frac{2f_i'}{f_i - 1}\right) - \left(\frac{f_j''}{f_j'} - \frac{2f_j'}{f_j - 1}\right),$$

where $1 \leq i < j \leq q$. If $\Delta_{ij} \not\equiv 0$, let z_0 be a simple zero of $f_i(z) - 1$ and $f_j(z) - 1$, then it is easy to see that z_0 is a zero of Δ_{ij}. Denote by $\overline{N}_{(2}(r, \frac{1}{f_k - 1})$ the reduced counting function of the zeros of $f_k(z) - 1$ with multiplicities ≥ 2. Then, by Lemma 3.2.1 and 3.2.2, we have

$$\begin{aligned}
\overline{N}_0(r,1,f_i,f_j) &\leq N(r, \tfrac{1}{\Delta_{ij}}) + \overline{N}_{(2}(r, \tfrac{1}{f_i-1}) + \overline{N}_{(2}(r, \tfrac{1}{f_j-1}) \\
&\leq T(r,\Delta_{ij}) + O(1) + S(r,f_i) + S(r,f_j) \\
&\leq N(r,\Delta_{ij}) + S(r,f_i) + S(r,f_j) \\
&\leq \overline{N}(r, \tfrac{1}{f_i-1}) - \overline{N}_0(r,1,f_i,f_j) + \overline{N}(r, \tfrac{1}{f_j-1}) - \overline{N}_0(r,1,f_i,f_j) + S(r,f_i) + S(r,f_j) \\
&\leq T(r,f_i) + T(r,f_j) - 2\overline{N}_0(r,1,f_i,f_j) + S(r,f_i) + S(r,f_j).
\end{aligned}$$

Therefore,

$$3\overline{N}_0(r,1) \leq 3\overline{N}_0(r,1,f_i,f_j) \leq T(r,f_i) + T(r,f_j) + S(r,f_i) + S(r,f_j).$$

Now, assume that $q = 2n$ is even. If $\Delta_{ij} \equiv 0$ and $\Delta_{ik} \equiv 0$ for $j \neq k$, then, by Lemma 3.2.3, we get $f_j \equiv f_k$ which is impossible by assumption. Therefore, there are at most n of Δ_{ij} which are identically zero and we may assume that only Δ_{12}, $\Delta_{34}, \ldots$, $\Delta_{(q-1)q}$ may be identically zero. Apply the above inequality to all Δ_{ij} which are nonzero and add together, we obtain

$$\left(\binom{q}{2} - \frac{q}{2}\right) 3\overline{N}_0(r,1) \leq (q-2)\sum_{j=1}^{q} T(r,f_j) + \sum_{j=1}^{q} S(r,f_j)$$

Hence,

$$\tau \leq \frac{2n-2}{3[n(2n-1)-n]} = \frac{1}{3n} = \frac{2}{3q}.$$

Finally, we assume that $q = 2n+1$ is odd. By the same argument as above, we may assume that only Δ_{12}, $\Delta_{34}, \ldots$, $\Delta_{(q-2)(q-1)}$ may be identically zero and obtain the following inequality

$$\left(\binom{q}{2} - \frac{q-1}{2}\right) 3\overline{N}_0(r,1) \leq (q-2)\sum_{j=1}^{q-1} T(r,f_j) + (q-1)T(r,f_q) + \sum_{j=1}^{q} S(r,f_j).$$

Since

$$\overline{N}_0(r,1) \leq \overline{N}(r, \frac{1}{f_j - 1}) \leq T(r,f_j) + O(1),\ 1 \leq j \leq q-1,$$

we have

$$(q-1)\overline{N}_0(r,1) \leq \sum_{j=1}^{q-1} T(r,f_j) + O(1).$$

Combine these inequalities, we have

$$\left\{3\left(\binom{q}{2} - \frac{q-1}{2}\right) + (q-1)\right\}\overline{N}_0(r,1) \leq (q-1)\sum_{j=1}^{q} T(r,f_j) + \sum_{j=1}^{q} S(r,f_j).$$

Therefore,

$$\tau \leq \frac{2n}{3[n(2n+1)-n]+2n} = \frac{1}{3n+1} = \frac{2}{3q-1}.$$

❑

Obviously, Theorem 3.1.3 generalizes Theorem 3.1.2. An easy consequence of Theorem 3.1.3 is the following corollary.

Corollary 3.3.1 *Let $f_j \in \mathcal{A}$, $1 \leq j \leq q$, be distinct, where $q \geq 3$. If $\tau > \frac{2}{3q}$ when q is even or $\tau > \frac{2}{3q-1}$ when q is odd, then at least two of f_j are the same.*

The inequality in the main theorem is sharp for $q = 3, 4$. When $q = 3$, the example can be found in [14]. When $q = 4$, let f_1, f_2, f_3, f_4 be the following functions

$$f_1(z) = e^z, f_2(z) = e^{-z}, f_3(z) = e^{2z}, \text{and } f_4(z) = e^{-2z}. \tag{3.3.1}$$

Clearly, they are meromorphic functions of class $\mathcal{A}$ and we have

$$\overline{N}_0(r, 1) = \overline{N}(r, \frac{1}{f_1 - 1}) = T(r, f_1) + S(r, f_1),$$

where the first equality follows from the definition of f_j, $1 \leq j \leq 4$, and the second one follows from Lemma 3.2.2. Moreover,

$$T(r, f_2) = T(r, f_1)+O(1), \; T(r, f_3) = 2T(r, f_1)+O(1), \text{ and } T(r, f_4) = 2T(r, f_1)+O(1).$$

Therefore,

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r, 1)}{\sum_{j=1}^{4} T(r, f_j)} = \lim_{r\to\infty} \frac{T(r, f_1) + S(r, f_1)}{6T(r, f_1) + O(1)} = \frac{1}{6}.$$

3.4 A Conjecture

Our main result Theorem 3.1.3 says that any q distinct meromorphic functions of class $\mathcal{A}$ must satisfies

$$\begin{cases} \tau \leq \frac{2}{3q} & \text{if } q \text{ is even,} \\ \tau \leq \frac{2}{3q-1} & \text{if } q \text{ is odd.} \end{cases}$$

For $q = 3, 4$, this result is sharp. But for $q \geq 5$, we don't know whether it is sharp or not. As the construction of the example (3.3.1), we can follow exact the same pattern to construct the following examples for $q \geq 5$:

$$f_1(z) = e^z, f_2(z) = e^{-z}, \ldots, f_{2n-1}(z) = e^{nz}, f_{2n}(z) = e^{-nz} \text{ if } q = 2n,$$

and

$$f_1(z) = e^z, f_2(z) = e^{-z}, \ldots, f_{2n}(z) = e^{-nz}, f_{2n+1} = e^{(n+1)z} \text{ if } q = 2n + 1.$$

Apply the same arguments as above, we obtain that

$$\tau = \begin{cases} \frac{4}{q(q+2)} & \text{if } q \text{ is even,} \\ \frac{4}{(q+1)^2} & \text{if } q \text{ is odd.} \end{cases}$$

The numbers τ match Theorem 3.1.3 in the cases $q = 3, 4$, but less than the numbers there. Therefore, it is reasonable to conjecture that the examples actually provide the sharp conditions.

Conjecture. Let $f_1, f_2, \ldots, f_q$ be q distinct meromorphic functions of class $\mathcal{A}$, where $q \geq 3$. Then

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{4}{q(q+2)}$$

when q is even, and

$$\tau = \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{4}{(q+1)^2}$$

when q is odd.