2 The Vertex Set of Uniquely Colorable (l,m)-Uniform Mixed Hypergraphs

Definition 2.1 [1] *A proper i-coloring of a mixed hypergraph $\mathcal{H}$ is called a strict i-coloring, if each of the i colors is used.*

Definition 2.2 [2] *A mixed hypergraph $\mathcal{H}$ is called uniquely colorable(u.c hypergraph or u.c for short)if it has precisely one strict coloring apart from permutation of colors. The class of uniquely colorable mixed hypergraphs is denoted by $\mathcal{UC}$.*

Definition 2.3 *For $2 \leq l,\ m \leq n = |X|$ let $K(n,l,m) = (X,\mathcal{C},\mathcal{D}) = (X,\binom{X}{l},\binom{X}{m})$. Hence, $|C| = \binom{n}{l}$ and $|D| = \binom{n}{m}$. We call $K(n,l,m)$ the complete (l,m)-uniform mixed hypergraph of order n.*

Theorem 2.4 [3] *$K(n,l,m)$ is uncolorable if and only if $n \geq (l-1)(m-1)+1$.*

Lemma 2.5 *If $K(n,l,m)$ is colorable, then $K(n,l,m)$ is not uniquely colorable.*

Proof.

If $K(n,l,m)$ is colorable, then we can find a coloring of $K(n,l,m)$ whose feasible partition $c = X_1 \cup X_2 \cup \cdots \cup X_t$. Since $\mathcal{C} = \binom{X}{l}$ and $\mathcal{D} = \binom{X}{m}$, $t \leq (l-1)$ and $|X_i| \leq (m-1)$ for all i. WLOG, we assume that $|X_i| \geq |X_j|$ if $i < j$. We redistributed the elements of each part of the feasible partition such that $|X_i'| = |X_i|$ for all i, and there exists an i such that $X_i' \neq X_j$ for all j. Then $X_1' \cup X_2' \cup \cdots \cup X_t'$ is a feasible partition of H. Hence, $K(n,l,m)$ is not uniquely colorable.

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Definition 2.6 *In a mixed hypergraph $\mathcal{H} = (X,\mathcal{C},\mathcal{D})$, a mixed hypergraph $\mathcal{H}'$ is called a subhypergraph of $\mathcal{H}$ if $\mathcal{H}' = (X,\mathcal{C}',\mathcal{D}')$ where $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{D}' \subseteq \mathcal{D}$.*

Lemma 2.7 *If a mixed hypergraph $\mathcal{H}$ is colorable, but not uniquely colorable, then neither is any subhypergraph of $\mathcal{H}$.*

Proof.

$\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is colorable, but not uniquely colorable. Therefore, there exist at least two strict colorings of $\mathcal{H}$, c_1 and c_2. Hence, for all $\mathcal{C}$-edges and $\mathcal{D}$-edges are colorable by c_1 and c_2. Let $\mathcal{H}' = (X, \mathcal{C}', \mathcal{D}')$ be a subhypergraph of $\mathcal{H}$. If $C \in \mathcal{C}'$, then $C \in \mathcal{C}$. Hence, C is colorable by c_1 and c_2. Similarly, D is colorable by c_1 and c_2 if $D \in \mathcal{D}'$. Therefore, c_1 and c_2 are strict colorings of $\mathcal{H}'$. Hence, $\mathcal{H}'$ is not u.c.

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Theorem 2.8 *The mixed hypergraph* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *with* $|X| = n$ *is u.c if and only if* $n \geq (l-1)(m-1)+1$.

Proof.

Necessity. We assume $n < (l-1)(m-1)+1$. By Theorem 2.4, Lemma 2.5, and Lemma 2.7, $K(n,l,m)$ is colorable, and the subhypergraph $\mathcal{H}$ of $K(n,l,m)$ is colorable, but not u.c.

Sufficiency. If $n \geq (l-1)(m-1)+1$. We consider that

$$c(x) = \bigcup_{i=1}^{\chi} X_i \ , \ \ |X_i| = (m-1) \ \ \forall\, i = 1, 2, \ \ldots\ , (\chi - 1),$$

$$|X_\chi| = \begin{cases} (m-1) & \text{if } (m-1) \mid n, \\ n \ mod \ (m-1) & \text{if } (m-1) \nmid n. \end{cases}$$

Let $\mathcal{D} = \binom{X}{m}$ and $\mathcal{C} = \binom{X}{l} - \bigcup_j E_j$ where $|E_j \cap X_i| \leq 1$ and $|E_j| = l$ for all i, j.

Since $|D| = m$ for all $D \in \mathcal{D}$ and $|X_i| \leq m-1$ for $i = 1, 2, \ldots, \chi$, there exist $x_i, x_j \in D$ such that $c(x_i) \neq c(x_j)$. Since $|C| = l$ for all $C \in \mathcal{C}$ and $\mathcal{C} = \binom{X}{l} - \bigcup_j E_j$, there exist $x_s, x_t \in C$ such that $c(x_s) = c(x_t)$. Hence, c is a strict coloring of $\mathcal{H}$.

We consider $c' \neq c$. We must prove that c' is not a strict coloring of $\mathcal{H}$. Let $c' = X_1' \cup X_2' \cup \ \cdots \ \cup X_t'$ be a feasible partition. If $\mathcal{D} = \binom{X}{m}$, then $|X_i'| \leq (m-1)$ for $i = 1, 2, \ldots, t$. Therefore, $t \geq \chi$. Since $c' \neq c$, there exist $x_i, x_j \in X$ such that $c(x_i) = c(x_j) = k$, but $c'(x_i) \neq c'(x_j)$, $c(x_i) = c(x_j) = k$ and $\mathcal{C} = \binom{X}{l} - \bigcup_j E_j$ where $l \leq \chi \leq t$. Therefore, there exists a $\mathcal{C}$-edge $C_0 \in \mathcal{C}$ such that $|C_0 \cap X_s| \leq 1$ except $s = k$ and $|C_0 \cap X_s'| \leq 1$ for all s. It contradicts to the definition of $\mathcal{C}$-edge. Hence, $\mathcal{H}$ is u.c.

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