# 3 The Uniquely Colorable $r$-Uniform Hypergraphs

**Definition 3.1** *For a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, the partial mixed subhypergraph $\mathcal{H} = (X, \mathcal{C}, \phi)$, also denoted by $\mathcal{H}_{\mathcal{C}} = (X, \mathcal{C})$ is called a $\mathcal{C}$-hypergraph, and the partial mixed subhypergraph $\mathcal{H} = (X, \phi, \mathcal{D})$, also denoted by $\mathcal{H}_{\mathcal{D}} = (X, \mathcal{D})$ is called a $\mathcal{D}$-hypergraph.*

**Definition 3.2** *In a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, an alternating sequence*

$$x_0 C_0 x_1 C_1 x_2 \ldots x_{t-1} C_{t-1} x_t$$

*of distinct vertices, which the possible exception of $x_0 = x_t$, and possibly repeated edges satisfying*

$$x_i, x_{i+1} \in C_i, \ i = 0, 1, \ldots, t-1,$$

*is called a $\mathcal{C}$-path connecting the vertices $x_0$ and $x_t$.*

**Definition 3.3** *A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is $\mathcal{C}$-connected if for any pair of its vertices there is a $\mathcal{C}$-path connecting them.*

In the first instance, we discuss an $r$-uniform $\mathcal{C}$-hypergraph and construct a uniquely colorable $r$-uniform $\mathcal{C}$-hypergraph. Then we discuss an $r$-uniform $\mathcal{D}$-hypergraph. Moreover, we construct a uniquely colorable $r$-uniform $\mathcal{D}$-hypergraph.

**Theorem 3.4** *If $\mathcal{H}$ is an $r$-uniform $\mathcal{C}$-hypergraph, then*

*(1) if $r = 2$ and $\mathcal{H}$ is $\mathcal{C}$-connected, then $\mathcal{H}$ is u.c;*

*(2) if $r > 2$, then $\mathcal{H}$ is not a uniquely colorable $\mathcal{C}$-Hypergraph.*

**Proof.**

(1) If $\mathcal{H}$ is $\mathcal{C}$-connected, then there exists a $\mathcal{C}$-path connecting for any pair of vertices. Since $\mathcal{H}$ is a 2-uniform $\mathcal{C}$-hypergraph, $c(x) = c(y)$ for all $x, y \in X$. Hence, $\chi(\mathcal{H}) = \bar{\chi}(\mathcal{H}) = 1$ and $R(\mathcal{H}) = (1, 0, \ldots, 0)$. Therefore, $\mathcal{H}$ is u.c.

(2) Let

$$c_1(x) = 1 \quad , x \in X.$$

Then $c_1$ is a strict coloring of all $\mathcal{C}$-hypergraphs.

$$c_2(x) = \begin{cases} 1 & \text{if } x \neq x_n, \\ 2 & \text{if } x = x_n. \end{cases}$$

Then $c_2 = X_1 \cup X_2$, where $X_1 = X - \{x_n\}$ and $X_2 = \{x_n\}$. There exist $x, y \in C$ such that $c_2(x) = c_2(y) = 1$ for all $C \in \mathcal{C}$. Hence, any $\mathcal{C}$-edge is colorable by $c_2$. Therefore, $c_1$ and $c_2$ are strict colorings of $\mathcal{H}$. Hence, $\mathcal{H}$ is not u.c.

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**Algorithm 3.5** *Let* $C = \{C_i | i = 1, 2, \ldots, (n-1)\}$ *where* $C_i = \{x_i, x_{i+1}\}$. *Then* $\mathcal{H} = (X, \mathcal{C})$ *is a u.c 2-uniform* $\mathcal{C}$*-hypergraph.*

Since $C = \{C_i | i = 1, 2, \ldots, (n-1)\}$ where $C_i = \{x_i, x_{i+1}\}$, then $\mathcal{H}$ is $\mathcal{C}$-connected. By Theorem3.4, $\mathcal{H}$ is a u.c 2-uniform $\mathcal{C}$-hypergraph.

**Theorem 3.6** *If* $\mathcal{H}$ *is a 2-uniform* $\mathcal{C}$*-hypergraph with* $|\mathcal{C}| < (n-1)$, *then* $\mathcal{H}$ *is not u.c.*

**Proof.**

Since $|\mathcal{C}| < (n-1)$ and $|X| = n$, then there exist at least two components. We assume $\mathcal{H}$ has $k$ components for $k \geq 2$. We collect vertices from the $j^{th}$ component, denoted by $X_j$. Let

$$c_1(x) = 1 \quad , x \in X;$$

$$c_2(x) = \begin{cases} 1 & \text{if } x \notin X_k, \\ 2 & \text{if } x \in X_k. \end{cases}$$

Then $c_2 = X_1' \cup X_2'$ where $X_1' = X - X_k$ and $X_2' = X_k$. There exists a $j$ such that $C \subseteq X_j$ for any $\mathcal{C}$-edge. If $j = k$, then $C \subseteq X_2'$. Otherwise $C \subseteq X_1'$. Therefore, $c_1$ and $c_2$ are strict colorings of $\mathcal{H}$. Hence, $\mathcal{H}$ is u.c.

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**Corollary 3.7** *If* $\mathcal{H}$ *is a u.c 2-uniform* $\mathcal{C}$*-hypergraph, then* $\mathcal{H}$ *has at least* $n-1$ $\mathcal{C}$*-edges.*

**Theorem 3.8** *If* $\mathcal{H}$ *is an* $r$*-uniform* $\mathcal{D}$*-hypergraph, then*

*(1) if* $r = 2$ *and* $\mathcal{D} = \binom{X}{2}$, *then* $\mathcal{H}$ *is u.c;*

*(2) if* $r > 2$, $\mathcal{H}$ *is not a uniquely colorable* $\mathcal{D}$*-Hypergraph.*

**Proof.**

(1) A 2-uniform $\mathcal{D}$-hypergraph $\mathcal{H}$ is a simple graph. And a u.c simple graph necessarily has $\binom{n}{2}$ edges. Hence, $\mathcal{H}$ is u.c if $\mathcal{D} = \binom{X}{2}$.

(2) Let

$$c_1(x_i) = i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Then $c_1$ is a strict coloring of all $\mathcal{D}$-hypergraphs. Let

$$c_2(x_i) = \begin{cases} i & \text{if } i \neq n, \\ n-1 & \text{if } i = n. \end{cases}$$

Then $c_2 = X_1 \cup X_2 \cup \cdots \cup X_{n-1}$ where $X_i = \{x_i\}$ for $i = 1, 2, \ldots, (n-2)$, and $X_{n-1} = \{x_{n-1}, x_n\}$. There exist $x, y \in D$ such that $c(x) \neq c(y)$ for any $\mathcal{D}$-edge. Therefore, $c_1$ and $c_2$ are strict colorings of $\mathcal{H}$. Hence, $\mathcal{H}$ is not u.c.

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We have the following Algorithm to show the construction of a u.c 2-uniform $\mathcal{D}$-hypergraph.

**Algorithm 3.9** *Let* $\mathcal{D} = \binom{X}{2}$. *Then* $\mathcal{H} = (X, \mathcal{D})$ *is a u.c* 2*-uniform* $\mathcal{D}$*-hypergraph.*

**Corollary 3.10** *If* $\mathcal{H}$ *is a u.c* 2*-uniform* $\mathcal{D}$*-hypergraphs, then* $\mathcal{H}$ *has at least* $\binom{X}{2}$ $\mathcal{D}$*-edges.*