4 Constructions of Uniquely Colorable Uniform Mixed Hypergraphs

We have constructed two u.c r-uniform $\mathcal{C}$- and $\mathcal{D}$-hypergraphs. In this chapter, we will construct u.c uniform mixed hypergraphs by two ways. We must construct u.c $(2, m)$-uniform mixed hypergraphs and $(3, m)$-uniform mixed hypergraphs first before we construct u.c uniform mixed hypergraphs.

Algorithm 4.1 *Let $n \geq m$ and $X = \{x_1, x_2, \ldots, x_n\}$. And $\mathcal{H}$ is a $(2, m)$-uniform mixed hypergraph.*

1. *Let $\mathcal{D} = \binom{X}{m}$ and $\mathcal{C} = \{C_i | \ i = 1, 2, \ldots, (n-1)\} - \{C_i | \ (m-1) \mid i\}$ where $C_i = \{x_i, x_{i+1}\}$ for $i = 1, 2, \ldots, (n-1)$.*
2. *Redenote indices of $\mathcal{C}$-edges. Let new $\mathcal{C} = \{C_j | j = 1, 2, \ldots, k\}$ where*

$$k = \begin{cases} \dfrac{(m-2)n}{(m-1)} & \text{if } (m-1) \mid n, \\ \\ (m-2)\lfloor \dfrac{n}{(m-1)} \rfloor + n \bmod (m-1) - 1 & \text{if } (m-1) \nmid n. \end{cases}$$

Then $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a u.c $(2, m)$-uniform mixed hypergraph.

Theorem 4.2 *If $\mathcal{H}$ is a $(2, m)$-uniform mixed hypergraph satisfy Algorithm 4.1, then $\mathcal{H}$ is u.c.*

Proof.

In order to prove this theorem clearly, we use the form of $\mathcal{C}$-edges by Algorithm 4.1 step1. So $\mathcal{C} = \{C_i | \ i = 1, 2, \ldots, (n-1)\} - \{C_i | \ (m-1) \mid i\}$. If $(m-1) \mid i$, then $\{x_i, x_{i+1}\} \notin \mathcal{C}$. Hence, k in Algorithm 4.1 have two different values.

Case1. $(m-1) \mid n$.

Since $\mathcal{C} = \{C_i | \ i = 1, 2, \ldots, (n-1)\} - \{C_i | \ (m-1) \mid i\}$, $\mathcal{H}_C = (X, \mathcal{C})$ has $n/(m-1)$ components, denoted by X_i for $i = 1, 2, \ldots, n/(m-1)$.

Let $c(i) = X_i$ for all i. We have $|X_i| = (m-1) < m$ for all i and $|D| = m$. Hence, any $\mathcal{D}$-edge $D \in \mathcal{D}$ is colorable by c. Therefore, c is a strict coloring of $\mathcal{H}$.

We consider $c' \neq c$, and $c' = X_1' \cup X_2' \cup \cdots \cup X_t'$. Since $|C| = 2$, it means that $t \leq n/(m-1)$. If $\{x_i, x_j\} \in \mathcal{C}$ then $c(x_i) = c(x_j)$ and $c'(x_i) = c'(x_j)$. Hence, there exists a j such that $X_i \subseteq X_j'$ for every j. Since $|X_i| = (m-1)$ for all i, $\left|X_j'\right| \geq (m-1)$. We have $\left|X_j'\right| \leq m-1$ for all j for $\mathcal{D} = \binom{X}{m}$. Hence, $\left|X_j'\right| = m-1$ and $c' = c$. It contradicts to our hypothesis. Therefore, $\mathcal{H}$ is u.c.

Case2. $(m-1) \nmid n$. If $(m-1) \nmid n$, then $\mathcal{H}_C = (X, \mathcal{C})$ has $\lceil n/(m-1) \rceil$ components, say X_i for $i = 1, 2, \ldots, \lceil n/(m-1) \rceil$ where $|X_i| = m-1$ for $i = 1, 2, \ldots, \lfloor n/(m-1) \rfloor$ and $\left|X_{\lceil n/(m-1) \rceil}\right| = n \ mod \ (m-1)$

Let $c(i) = X_i$ for all i. Since $|X_i| < m$ for all i and $|D| = m$, any $\mathcal{D}$-edge $D \in \mathcal{D}$ is colorable by c. Hence, c is a strict coloring of $\mathcal{H}$.

We consider $c' \neq c$, and $c' = X_1' \cup X_2' \cup \cdots \cup X_t'$. If $|C| = 2$, it means that $t \leq \lceil n/(m-1) \rceil$. If $\{x_i, x_j\} \in \mathcal{C}$, then $c(x_i) = c(x_j)$ and $c'(x_i) = c'(x_j)$. Hence, for all i there exists a j such that $X_i \subseteq X_j'$. Since $\mathcal{D} = \binom{X}{m}$, $\left|X_j'\right| \leq m-1$ for all j. For every i, there exists a j such that $X_i = X_j$. Therefore, $c' = c$ and it contradicts to hypothesis. Hence, $\mathcal{H}$ is u.c.

❏

Example 1 *We construct a u.c* $(2,4)$*-uniform mixed hypergraph with* $n = 8$*.*

Let $X = \{1, 2, \ldots, 8\}$. By Algorithm 4.1, let $\mathcal{D} = \binom{X}{4}$ and $\mathcal{C} = \{C_1,\ C_2,\ C_3,\ C_4,\ C_5\}$ where $C_1 = \{1,\ 2\}$, $C_2 = \{2,\ 3\}$, $C_3 = \{4,\ 5\}$, $C_4 = \{5,\ 6\}$, $C_5 = \{7,\ 8\}$. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. Hence, $c = \{1,\ 2,\ 3\} \cup \{4,\ 5,\ 6\} \cup \{7,\ 8\}$ is the strict coloring of $\mathcal{H}$. And $\mathcal{H}$ is a u.c $(2,4)$-uniform mixed hypergraph.

❏

From this example, we know that $\mathcal{D}$-edges do not necessarily equal to $\mathcal{D} = \binom{X}{4}$. We can reduce the number of $\mathcal{D}$-edges. Let the new $\mathcal{D}$-edges be $\{D_1,\ D_2,\ D_3\}$ where $D_1 = \{1,\ 2,\ 3,\ 4\}$, $D_2 = \{1,\ 2,\ 3,\ 7\}$, $D_3 = \{4,\ 5,\ 6,\ 7\}$. Since $\{1,\ 2,\ 3\}$ have a $\mathcal{C}$-path and so $\{1,\ 2,\ 3\}$ must be colored the same color. Similar to $\{4,\ 5,\ 6\}$, $\{7,\ 8\}$. Let $X_1 = \{1,\ 2,\ 3\}$, $X_2 = \{4,\ 5,\ 6\}$, and $X_3 = \{7,\ 8\}$. Since $D_1 = \{1,\ 2,\ 3,\ 4\}$, $D_2 = \{1,\ 2,\ 3,\ 7\}$, $D_3 = \{4,\ 5,\ 6,\ 7\}$, X_1, X_2, and X_3 have different colors. In fact, $\mathcal{C}$-edges make a $(2, m)$-uniform mixed hypergraph into a simple graph. Every component can be treated as a vertex. If the union of any two components have at least m vertices and $|D| = m$, then we can let all $\mathcal{D}$-edges contain exactly two components. Hence, a $(2, m)$-uniform mixed hypergraph with n vertices can be treated as a simple graph with k vertices. The uniquely colorable simple graphs must be complete graphs. Therefore, the number of $\mathcal{D}$-edges is $\binom{k}{2}$.

Lemma 4.3 *If $\mathcal{H} = (X, \mathcal{C}, \binom{X}{m})$ is a (2,m)-uniform mixed hypergraph and $\mathcal{H}_\mathcal{C}$ have two components whose number of vertices are at most $(m-1)$ and $m \geq 3$, then $\mathcal{H}$ is not u.c.*

Proof.

Suppose $\mathcal{H}_\mathcal{C}$ have k components, say X_i for $i = 1, \ldots, k$. WLOG, we assume $|X_i| \geq |X_j|$ if $i < j$. Then $|X_{k-1} \cup X_k| \leq (m-1)$ and $|X_i| \leq (m-1)$ for all i. Let $c_1(X_i) = i$. Since $D \in \binom{X}{m}$ and $|X_i| \leq (m-1)$, there exist $x, y \in D \ni c_1(x) \neq c_1(y)$ for all D. Hence, c_1 is a strict coloring of $\mathcal{H}$. We consider

$$c_2(X_i) = \begin{cases} i & \text{if } i \leq (k-1), \\ k & \text{if } i = k-1. \end{cases}$$

It means that $c_2 = X'_1 \cup X'_2 \cup \cdots \cup X'_{k-1}$ where $X'_i = X_i$ for $i = 1, 2, \ldots, (k-2)$ and $X'_{k-1} = X_{k-1} \cup X_k$. Since $|X'_i| \leq (m-1)$ for all i, there exist $x, y \in D$ such that $c_2(x) \neq c_2(y)$ for all $D \in \mathcal{D}$. Hence, c_2 is a strict coloring of $\mathcal{H}$. Therefore, $\mathcal{H}$ is not u.c.

❑

Definition 4.4 *Let $m_\mathcal{C}(n, l, m)$ be the minimum number of $\mathcal{C}$-edges of a u.c (l, m)-uniform mixed hypergraph. Let $m_\mathcal{D}(n, l, m)$ be the minimum number of $\mathcal{D}$-edges of a u.c (l, m)-uniform mixed hypergraph. And let $m^*_\mathcal{C}(n, l, m)$ be the minimum number of $\mathcal{C}$-edges among u.c (l, m)-uniform mixed hypergraph who has coloring that colors $m-1$ vertices by the first color.*

Theorem 4.5 *If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a u.c $(2, m)$-uniform mixed hypergraph and $m \geq 3$,* *then let*

$$m_C(n,2,m) = \begin{cases} \dfrac{(m-2)n}{(m-1)} & \textit{if } (m-1) \mid n, \\ \\ (m-2)\lfloor\dfrac{n}{(m-1)}\rfloor + n \ mod\ (m-1) - 1 & \textit{otherwise.} \end{cases}$$

Proof.

We have proven that $\mathcal{H}$ can be u.c for $m_C(n,2,m)$ $\mathcal{C}$-edges. Now, we prove that $\mathcal{H}$ is not u.c if $|\mathcal{C}| < m_C(n,2,m)$. Since $|\mathcal{C}| < m_C(n,2,m)$, there exist at least two components in $\mathcal{H}_C$ such that the vertices of them are at most $(m-1)$. By Lemma 4.3, $\mathcal{H}$ is not u.c.

❑

Lemma 4.6 *If $\mathcal{H}$ is a u.c (l,m)-uniform mixed hypergraph where $\mathcal{D} = \binom{X}{m}$, then*

$$m_c(n,l,m) = m_c^*(n,l,m) + m_c(n-(m-1), l-1, m).$$

Proof.

Let X_1 be the 1^{st} color class. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_\chi$ where $\mathcal{C}_i$ is the set of all $\mathcal{C}$-edges in which at least two vertices have the i^{th} color. If $C \in \mathcal{C}_i \cap \mathcal{C}_j$, then $C \in \mathcal{C}_i$ and $C \in \mathcal{C}_j$. Where $C \in \mathcal{C}_i$ implies that there are two vertices has the i^{th} color in C. Hence, there exist $x,\ y \in C$ such that $c(x) = c(y) = i$. Then the rest of the vertices of C can be colored any color. It contradicts to $C \in \mathcal{C}_j$. Then $\mathcal{C}_i \cap \mathcal{C}_j = \phi$ where $i < j$. Hence, $m_c(n,l,m) = m_c^*(n,l,m) + \sum_{i=2}^{\chi} min\,|\mathcal{C}_i|$. If every $C \in \mathcal{C}_i$ for $2 \le i \le \chi$, then $|C \cap X_1| \le 1$. Since $m_c(n,l,m)$ is minimum number, $|C \cap X_1| = 1$ for all $C \notin \mathcal{C}_1$. Hence, $|C|$ is reduced to $(l-1)$. Therefore, $\sum_{i=2}^{\chi} min\,|\mathcal{C}_i| = m_c(n-(m-1), l-1, m)$.

❑

Hence, we need only to construct the 1^{st} color class(X_1) of u.c (l,m)-uniform mixed hypergraphs. And we can get u.c (l,m)-uniform mixed hypergraphs by Lemma 4.6. Let X_i be the color classes where $|X_i| \ge |X_j|$ if $i < j$. In this construction, the size of all color classes is $(m-1)$ except for the last color class. And the size of the last color class is $n\ mod(m-1)$. Before we construct the 1^{st} color class of u.c (l,m)-uniform mixed hypergraphs, we construct a u.c $(3,m)$-uniform mixed hypergraph.

Algorithm 4.7 *Let* $n \geq (2m-1)$ *and* $X = \{x_1, x_2, \ldots, x_n\}$*. And* $\mathcal{H}$ *is a* $(3,m)$*-uniform mixed hypergraph.*

1. *Let* $\mathcal{D} = \binom{X}{m}$*; set* $i = 1,\ j = 1,\ \mathcal{C} = \emptyset$.
2. *Set* $C_{i,j} = \{x_i,\ x_{i+1},\ x_{j+i+1}\}$*; put* $\mathcal{C} := \mathcal{C} \cup \{C_{i,j}\}$.
3. *Put* $j := j+1$*; if* $j \leq (2m-2-i)$*, then go to step 2.*
4. *Put* $i := i+1$ *and* $j = 1$*; if* $i \leq m-2$*, then go to step 2.*
5. *Let* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$.

Then $\{x_1, x_2, \ldots, x_{m-1}\}$ *is the* 1^{st} *color class of a u.c* $(3,m)$*-uniform mixed hypergraph.*

Theorem 4.8 *If* $\mathcal{H}$ *is a* $(3,m)$*-uniform mixed hypergraph satisfy Algorithm 4.7, then* $\{x_1, x_2, \ldots, x_{m-1}\}$ *is the* 1^{st} *color class of* $\mathcal{H}$ *and is uniquely colored.*

Proof.

Suppose c is a strict coloring of $\mathcal{H}$. We prove that $\{x_1, x_2, \ldots, x_{m-1}\}$ is the 1^{st} color class(X_1) of coloring c by Mathematical Induction.

Basis step: If $i = 1$, $C_{1,j} = \{x_1,\ x_2,\ x_j\}$ for $j = 3, 4, \ldots, (2m-1)$. If $c(x_1) \neq c(x_2)$, then $c(x_j)$ is equal to $c(x_1)$ or $c(x_2)$ for $j = 3, 4, \ldots, (2m-1)$, by Pigeonhole Principle, $|X_{c(x_1)}|$ or $|X_{c(x_2)}|$ is greater than m. It is contradicted to $\mathcal{D} = \binom{X}{m}$. Hence, $c(x_1) = c(x_2)$.

Induction step: If $i = (k+1) \leq (m-2)$, $C_{k+1,j} = \{x_{k+1},\ x_{k+2},\ x_j\}$ for $j = (k+3), (k+4), \ldots, (2m-1)$. If $c(x_{k+1}) \neq c(x_{k+2})$, then $c(x_j)$ is equal to $c(x_{k+1})$ or $c(x_{k+2})$. And $c(x_i) = c(x_k + 1)$ for $i \leq k$. Hence, $|X_{c(x_{k+1})}|$ or $|X_{c(k+2)}|$ is greater than m. It contradicts to $\mathcal{D} = \binom{X}{m}$. Hence, $c(x_{k+1}) = c(x_{k+2})$. Then $\{x_1, x_2, \ldots, x_{m-1}\} \subseteq X_1$. $|X_1| \leq (m-1)$ for $\mathcal{D} = \binom{X}{m}$. Hence, $X_1 = \{x_1, x_2, \ldots, x_{m-1}\}$. We complete this proof.

❑

Example 2 *We construct a u.c* $(3,4)$*-uniform mixed hypergraph with* $n = 9$.

Let $X = \{1, 2, \ldots, 9\}$. Let $\mathcal{D} = \binom{X}{4}$ and $\mathcal{C} = \{C_1,\ C_2,\ \ldots\ ,\ C_{13}\}$ where

$$C_1 = \{1,\ 2,\ 3\},\ C_2 = \{1,\ 2,\ 4\},\ C_3 = \{1,\ 2,\ 5\},\ C_4 = \{1,\ 2,\ 6\},$$

$$C_5 = \{1,\ 2,\ 7\},\ C_6 = \{2,\ 3,\ 4\},\ C_7 = \{2,\ 3,\ 5\},\ C_8 = \{2,\ 3,\ 6\},$$

$$C_9 = \{2,\ 3,\ 7\},\ C_{10} = \{1,\ 4,\ 5\},\ C_{11} = \{1,\ 5,\ 6\},\ C_{12} = \{1,\ 7,\ 8\},$$

$$C_{13} = \{1,\ 8,\ 9\}.$$

In the example, $C_1, C_2, \ldots, C_9$ is from using the construction of Algorithm 4.7. Use Lemma 4.6, $C_{10}, C_{11}, \ldots, C_{13}$ is from using the construction of Algorithm 4.1. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. Hence, $c = \{1,\ 2,\ 3\} \cup \{4,\ 5,\ 6\} \cup \{7,\ 8,\ 9\}$ is the strict coloring of $\mathcal{H}$. And $\mathcal{H}$ is a u.c $(3, 4)$-uniform mixed hypergraph.

❑

Now, we construct the 1^{st} color class of u.c (l, m)-uniform mixed hypergraphs.

Algorithm 4.9 *Let* $n \geq 2 + [2(m-2)+1][\frac{(m-1)^{l-2}-1}{m-2}]$ *and* $X = \{x_1, x_2, \ldots, x_n\}$. *And* $\mathcal{H}$ *is an* (l, m)*-uniform mixed hypergraph.*

1. *Let* $\mathcal{D} = \binom{X}{m}$. *Set* $i = 1, j = 1, k = 1, t = 1$.
2. *Set* $E_{i,j}^{(0)} = \{x_i, x_{i+1}, x_{j+i+1}\}$.
3. *Put* $j := j + 1$*; if* $j \leq (2m - 2 - i)$*, then go to step 2.*
4. *Put* $i := i + 1$ *and* $j = 1$*; if* $i \leq m - 2$*, then go to step2; put* $i = 1$.
5. *Set* $s(i, k) = 1 + i + [2(m-2) + 2 - i][\frac{(m-1)^k - 1}{m-2}]$.
6. *Set* $C_{i,t}^{(k)} = E_{i,j}^{(k-1)} \cup \{x_{s(i,k)+t}\}$ *and* $E_{i,t}^{(k)} = C_{i,t}^{(k)}$.
7. *Put* $t := t + 1$*; if* $(m-1) \nmid (t-1)$*, then go to step 6; if* $t \leq (2m-2-i)(m-1)^k$*, then put* $j := j + 1$ *and go to step 6.*
8. *Put* $i := i + 1, j = 1, t = 1$*; if* $i \leq m - 2$*, then go to step 5.*
9. *Put* $k := k + 1, i = 1, j = 1, t = 1$*; if* $k \leq l - 3$*, then go to step 5.*
10. *Let* $\mathcal{C} = \{C_{i,t}^{(l-3)} \mid i = 1, 2, \ldots, m-2;\ t = 1, 2, \ldots, (2m-3)(m-1)^{(l-3)}\}$ *and* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$.

Then $\{x_1, x_2, \ldots, x_{m-1}\}$ *is the* 1^{st} *color class of a u.c* (l, m)*-uniform mixed hypergraph.*

Lemma 4.10 *If* $\mathcal{H}$ *is a* $(2, m)$*-uniform mixed hypergraph and uncolorable with* $\mathcal{D} = \binom{X}{m}$*, then* $\mathcal{H}$ *has at least* $(m-1)$ $\mathcal{C}$*-edges.*

Proof.

In the case, $\mathcal{H}$ has $(m-1)$ $\mathcal{C}$-edges. Let $C_i = \{x_1, x_{i+1}\}$ for $i = 1, 2, \ldots, (m-1)$. we claim that $\mathcal{H} = (X, \mathcal{C}, \binom{X}{m})$ is uncolorable. Suppose c is a strict coloring of $\mathcal{H}$. Hence, $c(x_{i+1}) = c(x_1)$ for $i = 1, 2, \ldots, (m-1)$. But $\mathcal{D} = \binom{X}{m}$ implies that the size of any color class is less then m. It contradicts to the hypothesis. Therefore, there exists no strict coloring of $\mathcal{H}$. Hence, $\mathcal{H}$ is uncolorable.

In the case, $\mathcal{H}$ has less than $(m-1)$ $\mathcal{C}$-edges. There exist at least two components in $\mathcal{H}_{\mathcal{C}}$. We assume $\mathcal{H}_{\mathcal{C}}$ has k components for $k \geq 2$, say X_i for $i = 1, 2, \ldots, k$. Hence, $c = \bigcup_{i=1}^{k} X_i$ is a partition of $\mathcal{H}$. And the size of every class is less than m. Therefore, any $\mathcal{D}$-edge D is colorable by c. Hence, $\mathcal{H}$ is colorable.

❑

In other word, there is a vertex belong to $\mathcal{C}$-edge which can not be colored. Now we begin to prove that $\mathcal{H}$ satisfy Algorithm 4.9, then $\{x_1, x_2, \ldots, x_{m-1}\}$ is the 1^{st} color class of $\mathcal{H}$.

Theorem 4.11 *If $\mathcal{H}$ is an (l, m)-uniform mixed hypergraph satisfy Algorithm 4.9, then $\{x_1, x_2, \ldots, x_{m-1}\}$ is the 1^{st} color class of $\mathcal{H}$ and is uniquely colored.*

Proof.

Suppose c is a strict coloring of $\mathcal{H}$. We must prove $\{x_1, x_2, \ldots, x_{m-1}\}$ is the 1^{st} color class X_1 of coloring c.

Basis step: If $l = 3$, then Algorithm 4.9 is equal to Algorithm 4.7. We have proven.

Induction step: If $l = k$, then $\mathcal{C} = \{C_{i,t}^{(k-3)} |$ for all $i,\ t\}$ and assumption is hold. And $C_{i,t}^{(k-3)} = E_{i,j}^{(k-3)}$. $l = k+1$, then $\mathcal{C} = \{C_{i,t}^{(k-2)} |$ for all $i,\ t\}$ where $C_{i,t}^{(k-2)} = E_{i,j}^{(k-3)} \cup \{x_{t+s}\}$. We contract the last two vertices of $\mathcal{C}$-edges to a new vertex y_j. Then $\mathcal{C}$-edges $E_{i,j}^{(k-3)} \cup \{x_{t+s}\}$ is contracted to a $\mathcal{C}$-edge $C_{i,j}^{\prime(k-3)}$. Also it satisfies the construction of Algorithm 4.9. Hence, $\{x_1, x_2, \ldots, x_{m-1}\} = X_1$ by assumption. By Lemma 4.10, there is a vertex not colored from construction of Lemma 4.10. Therefore, $C_{i,t}^{(k-2)} = C_{i,j}^{(k-3)} \cup \{x_{t+s}\}$. And we complete this proof.

❑

Example 3 *We construct a u.c* $(4,4)$*-uniform mixed hypergraph with* $n = 25$.

Let $X = \{1, 2, \ldots, 25\}$ and $\mathcal{D} = \binom{X}{4}$. By Algorithm 4.9, 4.7, 4.1, and Lemma 4.6. We list the result of the construction below.

$C_1 = \{1, 2, 3, 8\}$, $C_2 = \{1, 2, 3, 9\}$, $C_3 = \{1, 2, 3, 10\}$, $C_4 = \{1, 2, 4, 11\}$,

$C_5 = \{1, 2, 4, 12\}$, $C_6 = \{1, 2, 4, 13\}$, $C_7 = \{1, 2, 5, 14\}$, $C_8 = \{1, 2, 5, 15\}$,

$C_9 = \{1, 2, 5, 16\}$, $C_{10} = \{1, 2, 6, 17\}$, $C_{11} = \{1, 2, 6, 18\}$, $C_{12} = \{1, 2, 6, 19\}$,

$C_{13} = \{1, 2, 7, 20\}$, $C_{14} = \{1, 2, 7, 21\}$, $C_{15} = \{1, 2, 7, 22\}$, $C_{16} = \{2, 3, 4, 8\}$,

$C_{17} = \{2, 3, 4, 9\}$, $C_{18} = \{2, 3, 4, 10\}$, $C_{19} = \{2, 3, 5, 11\}$, $C_{20} = \{2, 3, 5, 12\}$,

$C_{21} = \{2, 3, 5, 13\}$, $C_{22} = \{2, 3, 6, 14\}$, $C_{23} = \{2, 3, 6, 15\}$, $C_{24} = \{2, 3, 6, 16\}$,

$C_{25} = \{2, 3, 7, 17\}$, $C_{26} = \{2, 3, 7, 18\}$, $C_{27} = \{2, 3, 7, 19\}$, $C_{28} = \{1, 4, 5, 6\}$,

$C_{29} = \{1, 4, 5, 7\}$, $C_{30} = \{1, 4, 5, 8\}$, $C_{31} = \{1, 4, 5, 9\}$, $C_{32} = \{1, 4, 5, 10\}$,

$C_{33} = \{1, 5, 6, 7\}$, $C_{34} = \{1, 5, 6, 8\}$, $C_{35} = \{1, 5, 6, 9\}$, $C_{36} = \{1, 5, 6, 10\}$,

$C_{37} = \{1, 4, 7, 8\}$, $C_{38} = \{1, 4, 8, 9\}$, $C_{39} = \{1, 4, 10, 11\}$, $C_{40} = \{1, 4, 11, 12\}$,

$C_{41} = \{1, 4, 13, 14\}$, $C_{42} = \{1, 4, 14, 15\}$, $C_{43} = \{1, 4, 16, 17\}$, $C_{44} = \{1, 4, 17, 18\}$,

$C_{45} = \{1, 4, 19, 20\}$, $C_{46} = \{1, 4, 20, 21\}$, $C_{47} = \{1, 4, 22, 23\}$, $C_{48} = \{1, 4, 23, 24\}$.

Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. Then $c = \{1, 2, 3\} \cup \{4, 5, 6\} \cup \{7, 8, 9\} \cup \{10, 11, 12\} \cup \{13, 14, 15\} \cup \{16, 17, 18\} \cup \{19, 20, 21\} \cup \{22, 23, 24\} \cup \{25\}$ is the strict coloring of $\mathcal{H}$.

❑

Corollary 4.12 *If* $\mathcal{H}$ *is a u.c* (l,m)*-uniform mixed hypergraph and* $m \geq 3$*, then*

$$m_c(n,l,m) = \begin{cases} \dfrac{3(m-1)[(m-1)^{(l-2)}-1]}{2} + \dfrac{n(m-2)}{(m-1)} - \\ (m-2)(l-2) & \text{if } (m-1) \mid n, \\ \\ \dfrac{3(m-1)[(m-1)^{(l-2)}-1]}{2} + (m-2)\lfloor \dfrac{n}{(m-1)} \rfloor - \\ (m-2)(l-2) + n \bmod (m-1) - 1 & \text{if } (m-1) \nmid n. \end{cases}$$

Proof.

By Theorem 4.8, we have

$$\begin{aligned} m^*_C(n,3,m) &= \sum_{k=0}^{m-3}[2(m-2)+1-k] \\ &= \frac{3(m-2)(m-1)}{2}. \end{aligned} \tag{4.1}$$

By Theorem 4.11, we have

$$\begin{aligned} m^*_C(n,l,m) &= \sum_{k=0}^{m-3}[2(m-2)+1-k](m-1)^{(l-3)} \\ &= \frac{3(m-2)(m-1)^{l-2}}{2}. \end{aligned} \tag{4.2}$$

By Lemma 4.6, Theorem 4.5, (4.1) and (4.2), we have

$$\begin{aligned} m_{\mathcal{C}}(n,l,m) &= m^*_{\mathcal{C}}(n,l,m)+m_{\mathcal{C}}(n-(m-1),l-1,m) \\ &= \sum_{i=0}^{l-3} m^*_{\mathcal{C}}(n-i(m-1),l-i,m)+m_{\mathcal{C}}(n-(l-2)(m-1),2,m) \\ &= \sum_{i=0}^{l-3} \frac{3(m-2)(m-1)^{(l-i)-2}}{2}+m_{\mathcal{C}}(n-(l-2)(m-1),2,m) \\ &= \frac{3(m-1)[(m-1)^{l-2}-1]}{2}+m_{\mathcal{C}}(n-(l-2)(m-1),2,m) \\ &= \begin{cases} \dfrac{3(m-1)[(m-1)^{(l-2)}-1]}{2}+\dfrac{n(m-2)}{(m-1)}- \\ (m-2)(l-2) & \text{if } (m-1) \mid n \\ \\ \dfrac{3(m-1)[(m-1)^{(l-2)}-1]}{2}+(m-2)\lfloor\dfrac{n}{(m-1)}\rfloor- \\ (m-2)(l-2)+n \ mod \ (m-1)-1 & \text{otherwise.} \end{cases} \end{aligned}$$

We complete this proof.

❏

We reduce the number of $\mathcal{C}$-edges in $\mathcal{H}$ and keep $\mathcal{H}$ being u.c in the foregoing constructions. But $n \geq 2+[2(m-2)+1][\frac{(m-1)^{l-2}-1}{m-2}]$ and $\mathcal{D} = \binom{X}{m}$, we use another way to construct $\mathcal{H}$ such that $n \geq (l-1)(m-1)+1$. Now, we reduce the number of $\mathcal{D}$-edges in $\mathcal{H}$ and keep $\mathcal{H}$ being u.c. First, we construct a u.c $(l,2)$-uniform mixed hypergraph.

Algorithm 4.13 *Let $n \geq l$ and $X = \{x_1, x_2, \ldots, x_n\}$. And $\mathcal{H}$ is an $(l,2)$-uniform mixed hypergraph.*

1. *Let $X' = \{x_1, x_2, \ldots, x_l\}$ and $\mathcal{D} = \binom{X'}{2}$; set $i = 1, j = 1, \mathcal{C} = \emptyset$.*
2. *Set $C_j^{(i)} = (X' - \{x_j\}) \cup \{x_{l+i}\}$; put $\mathcal{C} := \mathcal{C} \cup \{C_j^{(i)}\}$.*
3. *Put $j := j+1$; if $j \leq l-1$, then go to step 2.*
4. *Put $i := i+1$ and $j = 1$; if $i \leq n-l$, then go to step 2.*
5. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$.*

Then $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a u.c $(l,2)$-uniform mixed hypergraph.

Theorem 4.14 *If $\mathcal{H}$ is an $(l,2)$-uniform mixed hypergraph satisfy Algorithm 4.13, then $\mathcal{H}$ is u.c.*

Proof.

Suppose c is a strict coloring of $\mathcal{H}$. Since $\mathcal{D} = \binom{X'}{2}$, $c(x_i) \neq c(x_j)$ for $i, j = 1, 2, \ldots, l$. If $x_{l+t} \in X$ and $t \leq (n-l)$, then there exist $\mathcal{C}$-edges $C_j^{(t)} = (X - \{x_j\}) \cup \{x_{l+t}\}$ for $j = 1, 2, \ldots, (l-1)$. Consider $C_k^{(t)} = (X - \{x_k\}) \cup \{x_{l+t}\}$, if $c(x_l) \neq c(x_{l+t})$, there exist a vertex x_s such that $c(x_s) = c(x_{l+t})$ in $C_k^{(t)}$. But all vertices in $C_s^{(t)}$ color different colors for $C_s^{(t)} \in \mathcal{C}$, it contradicts to the definition of $\mathcal{C}$-edge. Hence, $c(x_l) = c(x_{l+t})$ for all $t \leq (n-l)$. Therefore, $c = \bigcup_{i=1}^{l} X_i$ where $X_i = \{x_i\}$ if $i = 1, 2, \ldots, l-1$ and $X_l = \{x_l, x_{l+1}, \ldots, x_n\}$ is the strict coloring of $\mathcal{H}$.

❑

Example 4 *We construct a u.c $(4,2)$-uniform mixed hypergraph with $n = 6$.*

Let $X = \{1, 2, \ldots, 6\}$ and $X' = \{1, 2, \ldots, 4\}$. Let $\mathcal{D} = \binom{X'}{2}$ and $\mathcal{C} = \{C_1,\ C_2, \ldots,\ C_6\}$ where

$$C_1 = \{2,\ 3,\ 4,\ 5\},\quad C_2 = \{1,\ 3,\ 4,\ 5\},\quad C_3 = \{1,\ 2,\ 4,\ 5\},\quad C_4 = \{2,\ 3,\ 4,\ 6\},$$

$$C_5 = \{1,\ 3,\ 4,\ 6\},\ C_6 = \{1,\ 2,\ 4,\ 6\}.$$

Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. Then $c = \{1\} \cup \{2\} \cup \{3\} \cup \{4,\ 5,\ 6\}$ is the strict coloring of $\mathcal{H}$. ❑

We only used four colors in example 4. By observing Algorithm 4.13 and example 4, we can discover that we only need l colors.

Theorem 4.15 *$\mathcal{H}$ is a u.c $(l,2)$-uniform mixed hypergraph and $l \geq 3$, then*

$$m_D(n,l,2) = \begin{cases} \binom{l}{2} - 1 & \text{if } n = l, \\ \\ \binom{l}{2} & \text{if } n > l. \end{cases}$$

Proof.

Case 1. $n = l$. We claim that $|\mathcal{D}| < \binom{l}{2} - 1$, then $\mathcal{H}$ is not u.c. There exist at least two pairs of vertices, say $\{x_n, x_{n-1}\}$, $\{x_{n-2}, x_s\}$, are not $\mathcal{D}$-edges for $|\mathcal{D}| < \binom{l}{2} - 1$, where x_s may equal x_n or x_{n-1}. If x_s equal x_n, then we re-denote x_n, x_{n-1}, such that x_s equal x_{n-1}. Otherwise, we denote $x_s = x_{n-3}$. Let

$$c_1(x_i) = \begin{cases} i & \text{if } i \neq n, \\ n-1 & \text{if } i = n. \end{cases}$$

If $x_s = x_{n-3}$, let

$$c_2(x_i) = \begin{cases} i & \text{if } i \leq s, \\ n-3 & \text{if } i = n-2, \\ n-2 & \text{if } i = n-1,\ n. \end{cases}$$

If $x_s = x_{n-1}$, let

$$c_2(x_i) = \begin{cases} i & \text{if } i \leq n-2, \\ n-2 & \text{if } i > n-2. \end{cases}$$

Therefore, c_1 and c_2 are strict colorings of $\mathcal{H}$. Hence, $\mathcal{H}$ is not u.c.

Case 2. $n > l$. We claim that $|\mathcal{D}| < \binom{l}{2}$, then $\mathcal{H}$ is not u.c. Since $|\mathcal{D}| < \binom{l}{2}$, $\chi(\mathcal{H}_D) < l$. Hence, $(l-1)$-coloring is a strict coloring of $\mathcal{H}_D$. We have $(l-1)$-coloring is a strict coloring of $\mathcal{H}$ for $|C| = l$. If $\chi(\mathcal{H}) < (l-1)$, then $\chi(\mathcal{H})$- and $(l-1)$-colorings are strict colorings of $\mathcal{H}$. Therefore, $\mathcal{H}$ is not u.c. Hence, we only consider $\chi(\mathcal{H}) = (l-1)$. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ and $c = X_1 \cup X_2 \cup \ldots \cup X_t$ where $t = (l-1)$ be a feasible partition of $\mathcal{H}$. Since $|\mathcal{D}| < \binom{l}{2}$ and $n > l$, there exists a vertex x_0 whose degree less than (l-1).

Case a. $0 \le deg(x_0) \le (l-3)$. Then we let

$$c'(x) = \begin{cases} c(x) & \text{if } x \neq x_0, \\ k & \text{if } x = x_0, \text{ where } k = min\{\{1,2,\ldots,(l-1)\} - c(\mathcal{N}(x_0) \cup \{x_0\})\}. \end{cases}$$

Hence, $\mathcal{H}$ is not u.c.

Case b. $deg(x_0) = (l-2)$. In other words, $deg(x) \ge (l-2)$, for all x. If $deg(x) \ge (l-2)$, then $(l+1)(l-2) \le \sum_{x \in X} deg(x) = 2|\mathcal{D}| \le 2(\binom{l}{2} - 1)$. It implies that $\sum_{x \in X} deg(x) = (l+1)(l-2)$. Therefore, $n = (l+1)$ and $deg(x) = (l-2)$ for all $x \in X$. Then $c = X_1 \cup X_2 \cup \cdots \cup X_{l-1}$ and $n = (l+1)$. In this case have two position. First, $|X_1| = 3$ and $|X_i| = 1$ for $i \ge 2$. Secondly, $|X_1| = |X_2| = 2$ and $|X_i| = 1$ for $i \ge 3$. By redenote the vertices, we can let $c = \{x_1, x_2, x_3\} \cup \{x_4\} \cup \cdots \cup \{x_{l+1}\}$ or $c = \{x_1, x_2\} \cup \{x_3, x_4\} \cup \{x_5\} \cup \cdots \cup \{x_{l+1}\}$. In first case, $c = \{x_1, x_2, x_3\} \cup \{x_4\} \cup \cdots \cup \{x_{l+1}\}$. Since $deg(x) = (l-2)$ for all i, $\{x_i, x_j\} \in \mathcal{D}$ for $i = 1,2,3$ and $4 \le j \le (l+1)$. And there exists $\{x_s, x_k\} \notin \mathcal{D}$. Let $c'(x_s) = c'(x_k)$, then $c' = \{x_1, x_2, x_3\} \cup \{x_s, x_t\} \cup \{x_4\} \cup \cdots \cup \{x_{l+1}\}$. Hence, c' is a strict $(l-2)$-coloring of $\mathcal{H}$. In the other case, $c = \{x_1, x_2\} \cup \{x_3, x_4\} \cup \{x_5\} \cup \cdots \cup \{x_{l+1}\}$, we have two positions. First, $\{x_i, x_3\}, \{x_i, x_4\} \in \mathcal{D}$ for $i = 1$ or 2 ; or $\{x_1, x_j\}, \{x_2, x_j\} \in \mathcal{D}$ for $j = 3$ *or* 4. Assume $\{x_1, x_3\}, \{x_1, x_4\} \in \mathcal{D}$, then there exists a vertex x_s such that $\{x_1, x_s\} \notin \mathcal{D}$. Let $c'(x_s) = c'(x_1) = c(x_s)$, then c' is a strict $(l-1)$-coloring of $\mathcal{H}$. Secondly, $\{x_i, x_3\} \in \mathcal{D}$ and $\{x_i, x_4\} \notin \mathcal{D}$ for $i = 1$ or 2, but not both; or $\{x_i, x_4\} \in \mathcal{D}$ and $\{x_i, x_3\} \notin \mathcal{D}$ for $i = 1$ or 2, but not both. Assume $\{x_1, x_3\} \in \mathcal{D}$ and $\{x_1, x_4\} \notin \mathcal{D}$, then $\{x_2, x_4\} \in \mathcal{D}$ and $\{x_2, x_3\} \notin \mathcal{D}$. Let $c' = \{x_1, x_4\} \cup \{x_2, x_3\} \cup \{x_5\} \cup \cdots \cup \{x_{l+1}\}$. Hence, c' is a strict $(l-1)$-coloring of $\mathcal{H}$. In the case, $\mathcal{H}$ is not u.c.

❑

Theorem 4.16 *If $\mathcal{H}$ is an r-uniform $\mathcal{D}$-hypergraph with $\chi(\mathcal{H}) = k$, then $\mathcal{H}$ has at least $\binom{(k-1)(r-1)+1}{r}$ edges.*

Proof.

We prove that the minimum numbers of $\mathcal{D}$-edges is $\binom{(k-1)(r-1)+1}{r}$ by Mathematical Induction.

Basis step: If $\chi(\mathcal{H}) = 2$, $|\mathcal{D}| = 1$ is true.

Induction step: If $\chi(\mathcal{H}) = k$, then we can find a k-coloring in which every vertices are colored by using the minimum number of colors. And we denote $c = X_1 \cup X_2 \cup \cdots \cup X_k$. Then $|X_1| \geq |X_2| \geq \ldots \geq |X_k| \geq 1$. In fact, $|X_{k-1}| \geq r-1$. If $|X_{k-1}| < r-1$, then let $X'_{k-1} = X_{k-1} \cup \{x\}$ and $X'_k = X_k - \{x\}$ where $x \in X_k$. Since $\left|X'_{k-1}\right| < r-1$, the new partition is a feasible partition of $\mathcal{H}$. It contradicts hypothesis of c. In order to find the minimum number of $\mathcal{D}$-edges, let $|X_1| = |X_2| = \ldots = |X_{k-1}| = r-1$, $|X_k| = 1$ by hypothesis of c and $|\mathcal{D}| = \binom{(k-1)(r-1)+1}{r}$. If $\chi(\mathcal{H}) = k+1$, $c = X_1 \cup X_2 \cup \cdots \cup X_k$ where $|X_1| = |X_2| = \ldots = |X_{k-1}| = r-1$ and $|X_k| = 1$ by assumption. Let $X = X_1 \cup X_2 \cup \cdots \cup X_k$ and $|X| = n$. In order to get a new color, we must add some vertices and edges such that $|X_k| = r-1$ and $|X_{k+1}| \geq 1$. Add x_{n+1} in X and let $X^{(1)} = X \cup \{x_{n+1}\}$. Since $c(x_{n+1}) = k$, $|D(x_{n+1})| = \binom{n}{r-1}$. If $|D(x_{n+1})| < \binom{n}{r-1}$, there exists a $V_1 \subset X^{(1)}$ and $|V_1| = r$ such that $V_1 \notin \mathcal{D}$. Now, let $c = \cup_{i=1}^{k} X'_i$ where $X'_1 = V_1$ and $|X'_i| = |X_i|$ for $i = 2, 3, \ldots, k$. And $\cup_{i=1}^{k} X'_i$ is a feasible partition. We reach a contradiction. Similarly, add x_{n+2} in X and let $X^{(2)} = X^{(1)} \cup \{x_{n+2}\}$. Since $c(x_{n+2}) = k$, $|D(x_{n+2})| = \binom{n+1}{r-1}$. Continuing these processes till x_{n+r-2}. Finally, we add x_{n+r-1} such that $c(x_{n+r-1}) = k+1$. Hence, $|D(x_{n+r-1})| = \binom{n+r-2}{r-1}$.

$$
\begin{aligned}
|\mathcal{D}| &= \binom{n}{r} + \binom{n}{r-1} + \binom{n+1}{r-1} + \cdots + \binom{n+r-2}{r-1} \\
&= \binom{r-1}{r-1} + \cdots + \binom{n-1}{r-1} + \binom{n}{r-1} + \cdots + \binom{n+r-2}{r-1} \\
&= \binom{n+r-1}{r} = \binom{k(r-1)+1}{r}.
\end{aligned}
$$

And we complete this proof.

❑

Now we construct a u.c (l, m)-uniform mixed hypergraph such that $n \geq (l-1)(m-1)+1$.

Algorithm 4.17 *Let* $n \geq (l-1)(m-1)+1$ *and* $X = \{x_1, x_2, \ldots, x_n\}$. *And* $\mathcal{H}$ *is an* (l, m)*-uniform mixed hypergraph.*

1. *Let* $X' = \{x_1, x_2, \ldots, x_{n'}\}$ *where* $n' = (l-1)(m-1)+1$ *and* $\mathcal{D} = \binom{X'}{m}$.

2. *Set* $E = \{\{x_{i_0},\ x_{i_1}, \ldots,\ x_{i_{(l-2)}},\ x_{n'}\} | \ 1 + k(m-1) \leq i_k \leq (k+1)(m-1)$ *for all* $k = 0, 1, \ldots, (l-2)\}$.

3. *Let* $\mathcal{C}_0 = \binom{X'}{l} - E$*; set* $\mathcal{C} = \mathcal{C}_0$.

4. *Set* $V = \{x_j | \ j = (k+1)(m-1)$ *for all* $k = 0, 1, \ldots, (l-2)\} \cup \{x_{n'}\}, i = 0, k = 0$.

5. *Set* $C_k^{(i)} = (V - \{x_j\}) \cup \{x_{n'+i}\}$ *where* $j = (k+1)(m-1)$*; put* $\mathcal{C} := \mathcal{C} \cup \{C_k^{(i)}\}$.

6. *Put* $k := k + 1$*; if* $k \leq l - 2$*, then go to step 5.*

7. *Put* $i := i + 1$*; if* $i \leq n - n'$*, then go to step 5.*

8. *Let* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$.

Then $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *is a u.c* (l, m)*-uniform mixed hypergraph.*

Theorem 4.18 *If* $\mathcal{H}$ *is an* (l, m)*-uniform mixed hypergraph satisfy Algorithm 4.17, then* $\mathcal{H}$ *is u.c.*

Proof.

Suppose c is a strict t-coloring of $\mathcal{H}$ and $c = X_1 \cup X_2 \cup \cdots \cup X_t$ is a feasible partition. Since $\mathcal{D} = \binom{X'}{m}$, $|X_i \cap X'| \leq (m-1)$ for all i. WLOG, there exists a k such that $X_i \cap X' \neq \emptyset$ for $i \leq k$ and $X_i \cap X' = \emptyset$ for $i > k$ after rearrangement of color classes. If $|X_i \cap X'| \leq (m-1)$ and $n' = (l-1)(m-1)+1$ for all i, then $k \geq l$. Since $\mathcal{C}_o \subseteq \mathcal{C}$, there exists a color class X_i such that $\{x_{1+(j-1)(m-1)},\ x_{2+(j-1)(m-1)}, \ldots,\ x_{j(m-1)}\} \subseteq X_i$ for $j = 1, 2, \ldots, (l-1)$. And we let $\{x_{1+(j-1)(m-1)},\ x_{2+(j-1)(m-1)}, \ldots,\ x_{j(m-1)}\} \subseteq X_j$ after rearrangement of color classes. Otherwise, there exists a vertex $x_{s+(t-1)(m-1)} \notin X_{(t-1)}$. By the assumption, $x_{s+(t-1)(m-1)} \in X_{i'}$ for some $i' \leq k$, since $k \geq l$ and $\mathcal{C}_o = \binom{X'}{l} - E$ where $E = \{\{x_{i_0},\ x_{i_1}, \ldots,\ x_{i_{(l-2)}},\ x_{n'}\} | \ 1 + k(m-1) \leq i_k \leq (k+1)(m-1)$ for all $k = 0, 1, \ldots, (l-2)\}$, there exists a $\mathcal{C}$-edge $C \in \mathcal{C}_o$ such that $|X_i \cap C| \leq 1$ for all i. It contradicts to the definition of $\mathcal{C}$-edge. Hence, $\{x_{1+j(m-1)},\ x_{2+j(m-1)}, \ldots,\ x_{(j+1)(m-1)}\} \subseteq X_j$ for $j = 0, 1, \ldots, (l-2)$ and $x_{n'} \in X_l$. If $C_k^{(i)} = (V - \{x_j\}) \cup \{x_{n'+i}\}$ for all $1 \leq i \leq (n-n')$ and $0 \leq k \leq (l-2)$, then $c(x_{n'+i}) = c(x_{n'})$. Hence, $c = \bigcup_{j=1}^{l} X_j$ where $X_j = \{x_{1+(j-1)(m-1)},\ x_{2+(j-1)(m-1)}, \ldots,\ x_{j(m-1)}\}$ for $j = 1, 2, \ldots, (l-1)$ and $X_l = \{x_{n'},\ x_{n'+1}, \ldots,\ x_n\}$ is the strict coloring of $\mathcal{H}$.

❑

Theorem 4.19 *If $\mathcal{H}$ is a u.c (l,m)-uniform mixed hypergraph with $|X| = (l-1)(m-1)+1$. Then $\mathcal{H}$ has at least $\binom{(l-1)(m-1)+1}{m} - 1$ $\mathcal{D}$-edges if $m = 2$ or $l = 3$. Otherwise, $\mathcal{H}$ has at least $\binom{(l-1)(m-1)+1}{m}$ $\mathcal{D}$-edges.*

Proof.

Case 1. $\mathcal{H}$ is an $(l,2)$-uniform mixed hypergraph with $|X| = l$ and is uc. We have proven in Theorem 4.15.

Case 2. $\mathcal{H}$ is a $(3,m)$-uniform mixed hypergraph with $|X| = 2m-1$ and is uc. Let $X = \{x_1, x_2, \ldots, x_{(2m-1)}\}$, $\mathcal{D} = \binom{X}{m} - \{x_m, x_{(m+1)}, \ldots, x_{(2m-1)}\}$, and $\mathcal{C} = \binom{X}{3}$. Let $c = X_1 \cup X_2$ where $X_1 = \{x_1, x_2, \ldots, x_{(m-1)}\}$ and $X_2 = \{x_m, x_{(m+1)}, \ldots, x_{(2m-1)}\}$. Since $|X_1| = m-1$ and $|X_2| = m$, there exist x_i, $x_j \in D$ such that $c(x_i) \neq c(x_j)$ for all $D \in \mathcal{D}$. Hence, c is a strict coloring of $\mathcal{H}$. Now we consider that c' is another strict coloring of $\mathcal{H}$. Since $\mathcal{C} = \binom{X}{3}$ and $\mathcal{D} = \binom{X}{m} - \{x_m, x_{(m+1)}, \ldots, x_{(2m-1)}\}$, then $c' = X_1' \cup X_2'$ where $|X_1'| = m-1$ and $|X_2'| = m$, or $|X_1'| = m$ and $|X_2'| = m-1$. WLOG, we assume that $|X_1'| < |X_2'|$. $X_1' = X_1$ and $X_2' = X_2$ for $\{x_m, x_{(m+1)}, \ldots, x_{(2m-1)}\} \notin \mathcal{D}$. By the definition of coloring, $c' = c$, we reach a contradiction. Hence, $\mathcal{H}$ is u.c by the construction. Now, we prove that $\mathcal{H}$ is not u.c if $|\mathcal{D}| = \binom{2m-1}{m} - 2$. It suffices to complete this proof. Consider $|\mathcal{D}| = \binom{2m-1}{m} - 2$. It means that $\mathcal{D} = \binom{X}{m} - \{V_1,\ V_2\}$ where $|V_1| = |V_2| = m$ and $V_1,\ V_2 \subseteq X$. Let $c = X_1 \cup X_2$ where $X_1 = V_1$ and $X_2 = (X - V_1)$, and $c' = X_1' \cup X_2'$ where $X_1' = V_2$ and $X_2' = (X - V_2)$. For all $D \in \mathcal{D}$, $D \cap X_i \neq \phi$ for $i = 1,2$, and $D \cap X_j' \neq \phi$ for $j = 1,2$, every $\mathcal{D}$-edge is colorable. And every $\mathcal{C}$-edge is colorable by c and c' for there are two colors in these colorings. Hence, c and c' are both strict colorings of $\mathcal{H}$. Therefore, $\mathcal{H}$ is not u.c. We complete this proof for $l = 3$.

Case 3. $\mathcal{H}$ is an (l,m)-uniform mixed hypergraph with $|X| = (l-1)(m-1)+1$ and is uc. Use Algorithm 4.17, we can get a u.c (l,m)-uniform mixed hypergraph $\mathcal{H}$. Now, we only need to show that $\mathcal{H}$ is not u.c if $|\mathcal{D}| = \binom{(l-1)(m-1)+1}{m} - 1$. Consider $|\mathcal{D}| = \binom{(l-1)(m-1)+1}{m} - 1$, it implies that $\mathcal{D} = \binom{X}{m} - \{V_1\}$ where $|V_1| = m$. WLOG, we assume that $V_1 = \{x_{(l-2)(m-1)+1}, x_{(l-2)(m-1)+2}, \ldots, x_{(l-1)(m-1)+1}\}$. Then let $c_1 = X_1 \cup X_2 \cup \cdots \cup X_{l-1}$, where

$$\begin{aligned}
X_1 &= \{x_1, x_2, \ \ldots \ , x_{m-1}\},\\
X_2 &= \{x_m, x_{m+1}, \ \ldots \ , x_{2(m-1)}\},\\
&\vdots\\
X_{l-2} &= \{x_{(l-3)(m-1)+1}, x_{(l-2)(m-1)+2}, \ \ldots \ , x_{(l-2)(m-1)}\},\\
X_{l-1} &= \{x_{(l-2)(m-1)+1}, x_{(l-2)(m-1)+2}, \ \ldots \ , x_{(l-1)(m-1)+1}\}.
\end{aligned}$$

And let $c' = X_1' \cup X_2' \cup \ \cdots \ \cup X_{l-1}'$, where

$$\begin{aligned}
X_1' &= \{x_1, x_{l-1}, \ \ldots \ , x_{(m-2)(l-2)+1}\},\\
X_2' &= \{x_2, x_l, \ \ldots \ , x_{(m-2)(l-2)+2}\},\\
&\vdots\\
X_{l-2}' &= \{x_{l-2}, x_{2(l-2)}, \ \ldots \ , x_{(l-2)(m-1)}\},\\
X_{l-1}' &= \{x_{(l-2)(m-1)+1}, x_{(l-2)(m-1)+2}, \ \ldots \ , x_{(l-1)(m-1)+1}\}.
\end{aligned}$$

Then every $\mathcal{D}$-edge is colorable by c and c'. And every $\mathcal{C}$-edge is also colorable by c and c' for there are $l-1$ colors in these colorings. Hence, $\mathcal{H}$ with $|\mathcal{D}| < \binom{(l-1)(m-1)+1}{m}$ is not u.c. We complete this proof.

❑

Theorem 4.20 *If $\mathcal{H}$ is a u.c (l,m)-uniform mixed hypergraph with $|X| > (l-1)(m-1)+1$, then $\mathcal{H}$ has at least $\binom{(l-1)(m-1)+1}{m}$ $\mathcal{D}$-edges.*

Proof.

We only need to show that $\mathcal{H}$ is not u.c for $|D| < \binom{(l-1)(m-1)+1}{m}$. Since $|D| < \binom{(l-1)(m-1)+1}{m}$, $\chi(\mathcal{H}) \le (l-1)$. Therefore, $(l-1)$-coloring is a strict coloring of $\mathcal{H}$ for $|C| = l$. If $\chi(\mathcal{H}) < (l-1)$, then $(l-1)$- and $\chi(\mathcal{H})$-colorings are strict colorings of $\mathcal{H}$. Hence, $\mathcal{H}$ is not u.c. Now, we consider $\chi(\mathcal{H}) = (l-1)$. Let $c = X_1 \cup X_2 \cup \cdots \cup X_{l-1}$ be a feasible partition of $\mathcal{H}$. WLOG, we assume that $|X_i| \ge |X_j|$ if $i < j$. If $|X_{l-1}| < m-1$, let $c' = \bigcup_{i=1}^{l-1} X_i'$ where $X_i' = X_i$ for $i = 1, 2, \ldots, (l-3)$, $X_{l-2}' = X_{l-2} - \{x\}$ and $X_{l-1}' = X_{l-1} \cup \{x\}$ for $x \in X_{l-2}$. Therefore, $c' = \bigcup_{i=1}^{l-1} X_i'$ is a feasible partition of $\mathcal{H}$ and $\mathcal{H}$ is not u.c. Hence, we consider $|X_i| \ge (m-1)$ for $i = 1, 2, \ldots, (l-1)$. If $|X_{l-2}| = |X_{l-1}| = m-1$, let $c' = \bigcup_{i=1}^{l-1} X_i'$ where $X_i' = X_i$ for $i = 1, 2, \ldots, (l-3)$, $X_{l-2}' = (X_{l-2} - \{x\}) \cup \{y\}$ and

$X'_{l-1} = (X_{l-1} - \{y\}) \cup \{x\}$ for $x \in X_{l-2}$ and $y \in X_{l-2}$. Therefore, $c' = \bigcup_{i=1}^{l-1} X'_i$ is a feasible partition of $\mathcal{H}$ and $\mathcal{H}$ is not u.c. Finally, we consider $|X_i| > (m-1)$ for $i = 1, 2, \ldots, (l-2)$ and $|X_{l-1}| \geq (m-1)$. Let $X_i = \{x_{1_i}, x_{2_i}, \ldots, x_{n_i}\}$ for $i = 1, 2, \ldots, (l-1)$.

$$\begin{aligned} X_1 &= \{x_{1_1}, x_{2_1}, \ldots, x_{(m-1)_1}, \ldots, x_{n_1}\}, \\ X_2 &= \{x_{1_2}, x_{2_2}, \ldots, x_{(m-1)_2}, \ldots, x_{n_2}\}, \\ &\vdots \\ X_{l-1} &= \{x_{1_{(l-1)}}, x_{2_{(l-1)}}, \ldots, x_{(m-1)_{(l-1)}}, \ldots, x_{n_{(l-1)}}\}. \end{aligned}$$

In order to let x_{k_i} for $k = 1, 2, \ldots, (m-1)$, $i = 1, 2, \ldots, (l-1)$ and x_{1_m} be u.c, we need have at least $\binom{(l-1)(m-1)+1}{m} - 1$ $\mathcal{D}$-edges. Since $n \geq (l-1)(m-1)+2$, $|\mathcal{D}| \geq \binom{(l-1)(m-1)+1}{m}$. Hence, we complete this proof.

❑

Example 5 *We construct a u.c $(3,3)$-uniform mixed hypergraph with $n = 6$.*

Let $X = \{1, 2, \ldots, 6\}$ and $X' = \{1, 2, \ldots, 5\}$. We list the result of the construction below. Let $\mathcal{D} = \binom{X'}{3}$ and $\mathcal{C} = \{C_1,\ C_2,\ \ldots,\ C_8\}$ where

$$C_1 = \{1,\ 2,\ 3\},\quad C_2 = \{1,\ 2,\ 4\},\quad C_3 = \{1,\ 2,\ 5\},\quad C_4 = \{1,\ 3,\ 4\},$$

$$C_5 = \{2,\ 3,\ 4\},\quad C_6 = \{3,\ 4,\ 5\},\quad C_7 = \{4,\ 5,\ 6\},\quad C_8 = \{2,\ 5,\ 6\},$$

Then $c = \{1,\ 2\} \cup \{3,\ 4\} \cup \{5,\ 6\}$ is the strict coloring of $\mathcal{H}$.

❑

Theorem 4.21 *If $\mathcal{H}$ is a u.c (l,m)-uniform mixed hypergraph, then*

$$m_{\mathcal{D}}(n,l,m) = \begin{cases} \binom{(l-1)(m-1)+1}{m} - 1 & \text{if } n = l \text{ and } m = 2, \text{ or } n = 2m-1 \text{ and } l = 3, \\ \binom{(l-1)(m-1)+1}{m} & \text{otherwise.} \end{cases}$$