4 Value Distribution of Meromorphic Functions in class $\mathcal{A}$ with Their Derivatives

In this section, we review the basic properties of meromorphic functions of class $\mathcal{A}$ and prove some results on the value distribution of such functions and their derivatives.

Definition 10 *A meromorphic function f is of class $\mathcal{A}$ if it satisfies*

$$\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})=S(r,f).$$

Remark. Meromorphic functions f of class $\mathcal{A}$ contain all meromorphic functions f satisfying either $\delta(0,f)=\delta(\infty,f)=1$ or $\Theta(0,f)=\Theta(\infty,f)=1$.

Lemma 4.1 *Let $f\in\mathcal{A}$ and $k\in\mathbb{N}$. Then*
(i) $T(r,\frac{f^{(k)}}{f})=S(r,f)$.
(ii) $T(r,f^{(k)})=T(r,f)+S(r.f)$.
(iii) $f^{(k)}\in\mathcal{A}$.

Proof. Since $f(z)\in\mathcal{A}$, we have $\overline{N}(r,\frac{1}{f})=S(r,f)$ and $\overline{N}(r,f)=S(r,f)$. Obviously,

$$\begin{aligned}T\left(r,\frac{f^{(k)}}{f}\right)&=N\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{f^{(k)}}{f}\right)\\&\le k\left\{\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right\}+S(r,f)\\&=S(r,f).\end{aligned}$$

So, (i) holds. Note that

$$T(r,f^{(k)})\le T\left(r,\frac{f^{(k)}}{f}\right)+T(r,f)\le T(r,f)+S(r,f),$$

and

$$\begin{aligned} T(r,f) &\le T(r,f^{(k)}) + T\left(r, \frac{f}{f^{(k)}}\right) \\ &= T(r,f^{(k)}) + T\left(r, \frac{f^{(k)}}{f}\right) + O(1) \\ &= T(r,f^{(k)}) + S(r,f). \end{aligned}$$

We get $T(r,f^{(k)}) = T(r,f) + S(r,f)$. Hence, (ii) holds. Finally, since

$$\begin{aligned} \overline{N}(r,f^{(k)}) &= \overline{N}(r,f) = S(r,f) = S(r,f^{(k)}), \\ \overline{N}\left(r, \frac{1}{f^{(k)}}\right) &\le \overline{N}\left(r, \frac{f}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f}\right) \\ &\le T\left(r, \frac{f}{f^{(k)}}\right) + S(r,f) \\ &= S(r,f) = S(r,f^{(k)}), \end{aligned}$$

We obtain $\overline{N}(r,f^{(k)}) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) = S(r,f^{(k)})$. That is (iii) ❑

Now, we can generalize Theorem 1.3 through Theorem 1.6 to the case of meromorphic functions of class $\mathcal{A}$ as follows.

Theorem 4.2 *If $f(z)$ is a transcendental entire function of class $\mathcal{A}$, k and n are two positive integers, then $f^{(k)}(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.*

Proof. Given a nonzero complex number a and consider the function

$$\psi(z) = \frac{f^{(k)}(z)f(z)^n}{a}.$$

By Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r,\psi) &\le \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + \overline{N}(r,\psi) + S(r,\psi) \qquad (4.1) \\ &= \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + S(r,\psi). \end{aligned}$$

By the definition of ψ and lemma 4.1, we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{\psi}\right) &\leq \overline{N}\left(r,\frac{1}{f}\right)+N_0\left(r,\frac{1}{f^{(k)}}\right) \\ &\leq N_0\left(r,\frac{1}{f^{(k)}}\right)+S(r,f) \\ &\leq S(r,f).\end{aligned} \tag{4.2}$$

Therefore, $\psi(z)=1$, that is, $f^{(k)}(z)f(z)^n=a$, has infinitely many roots. ❑

Theorem 4.3 *If $f(z)$ is a transcendental meromorphic function of class $\mathcal{A}$, k and n are two positive integers, then $f^{(k)}(z)f(z)^n$ assume all finite values except possibly zero infinitely often.*

Proof. Given a nonzero complex number a and consider the function

$$\psi(z)=\frac{f^{(k)}(z)f(z)^n}{a}.$$

As in the proof in Theorem 4.2, we have the following inequalities,

$$T(r,\psi)\leq \overline{N}\left(r,\frac{1}{\psi}\right)+\overline{N}\left(r,\frac{1}{\psi-1}\right)+\overline{N}\left(r,\psi\right)+S(r,\psi), \tag{4.3}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{\psi}\right) &\leq \overline{N}\left(r,\frac{1}{f}\right)+N_0\left(r,\frac{1}{f^{(k)}}\right) \\ &\leq S(r,f).\end{aligned} \tag{4.4}$$

By the definition of class $\mathcal{A}$,

$$\overline{N}(r,\psi)=\overline{N}(r,f)=S(r,f)$$

Therefore, $\psi(z)=1$, that is, $f^{(k)}(z)f(z)^n=a$, has infinitely many roots. ❑

Theorem 4.4 *Let $f(z)$ be a transcendental meromorphic function of class $\mathcal{A}$ and set*

$$\varphi(z)=f^{(k)}(z)-af(z)^n,$$

where k, $n\geq 3$ are integers and a is a nonzero complex number, Then $\varphi(z)$ assumes all finite values infinitely often.

Proof. Let b be an arbitrary complex number and consider the function

$$\psi(z) = \frac{f^{(k)} - b}{af(z)^n}.$$

Note that $S(r, \psi) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of finite linear measure.

Since the poles and zeros of $\psi(z)$ occur only at the zeros of $f^{(k)}(z) - b$ up to $S(r, f)$. If $f^{(k)}(z_0) - b = 0$, then z_0 is counted in $\overline{n}(r, \psi) + \overline{n}(r, \frac{1}{\psi})$ at most once. By lemma 4.1 and the definition of class $\mathcal{A}$, we have

$$\begin{aligned} \overline{N}(r, \psi) + \overline{N}\left(r, \frac{1}{\psi}\right) &\leq \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f) \\ &\leq T\left(r, f^{(k)}\right) + S(r, f) \\ &= T(r, f) + S(r, f). \end{aligned} \tag{4.5}$$

Then By Nevanlinna's second fundamental theorem and (4.5),

$$\begin{aligned} T(r, \psi) &\leq \overline{N}(r, \psi) + \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + S(r, \psi) \\ &\leq T(r, f) + S(r, f) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + S(r, \psi). \end{aligned} \tag{4.6}$$

On the other hand,

$$\begin{aligned} nT(r, f) &= T(r, f^n) \\ &= T\left(r, \frac{f^{(k)} - b}{a\psi}\right) \\ &\leq T\left(r, f^{(k)}\right) + T(r, \psi) + O(1) \\ &\leq T(r, f) + T(r, \psi) + O(1) \\ &= 2T(r, f) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + S(r, f). \end{aligned}$$

Therefore, we obtain

$$(n - 2)T(r, f) \leq \overline{N}\left(r, \frac{1}{\psi - 1}\right) + S(r, f).$$

Since $n \geq 3$,$\psi(z) = 1$, that is, $\varphi(z) = b$, has infinitely many roots. ❑

Theorem 4.5 *Let $f(z)$ be a transcendental meromorphic function of class $\mathcal{A}$ and $a_0(z), \dots, a_n(z)$ be small functions of f. Set*

$$\psi(z) = \sum_{i=0}^{n} a_i(z) f^{(i)}(z).$$

If $\delta(0, f) + \Theta(\infty, f) > 1$, then $\psi(z)$ assumes all finite values except possibly zero infinitely often or else $\psi(z)$ is identically constant.

Proof. Since f is a meromorphic function of class $\mathcal{A}$, we have

$$N\left(r, \frac{1}{f}\right) + \overline{N}(r, f) = S(r, f). \tag{4.7}$$

By Theorem 3.3, we get

$$\begin{aligned} T(r, f) &< \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r, f) \\ &< \overline{N}\left(r, \frac{1}{\psi - 1}\right) + S(r, f). \end{aligned}$$

Hence, $\psi(z) = 1$ has infinitely many roots. For arbitrary nonzero complex number w, the above proof shows that $\psi(z)/w = 1$ has infinitely many roots, that is, $\psi(z) = w$ has infinitely many roots. ❑