Chapter 1

Introduction

A *hypergraph* is a generalization of a graph in which any subset of a given set may be an edge rather than only two-element subsets [3]. A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set and each of $\mathcal{C}, \mathcal{D}$ is a collection of subsets of X, the $\mathcal{C}$-*edges* and $\mathcal{D}$-*edges*, respectively, and we assume that all $\mathcal{C}$-edges and $\mathcal{D}$-edges have at least two elements [3]. If a hypergraph $\mathcal{H}$ has no multiple edges and all its edges are of size k, then $\mathcal{H}$ is called a k-uniform hypergraph [3]. From the point of view, a simple graph is a 2-uniform hypergraph. For $1 \leq k \leq n$, we define the complete k-uniform hypergraph to be the hypergraph $\mathcal{K}_n^k = (X, D)$ such that $|X| = n$ and $\binom{X}{k}$ denotes all the k-subsets of X [3]. Similarly, so is the mixed hypergraph with complete $\mathcal{C}$-edges.

A *proper* λ-*coloring* of a mixed hypergraph is a function, c, from the vertex set to a set of λ colors so that each C-edge has two vertices with a common color and each D-edge has two vertices with distinct colors. A mixed hypergraph is λ-*colorable* if it has a proper coloring with at most λ colors. A *strict* λ-*coloring* is a proper λ-coloring using all λ colors, it means that the function, c, is onto [3].

In a colorable mixed hypergraph $\mathcal{H}$, the maximum (minimum) number of colors over all strict λ-colorings is called the *upper* (*lower*) *chromatic number* of $\mathcal{H}$ and is denoted by $\bar{\chi}(\mathcal{H})$ ($\chi(\mathcal{H})$) [4]. In a hypergraph $\mathcal{H} = (X, \mathcal{D})$, an alternating sequence $x_0 D_0 x_1 D_1 x_2 \ldots x_{t-1} D_{t-1} x_t$ of distinct vertices, with the possible exception of $x_0 = x_t$, and possibly repeated edges satisfying

$$x_i, x_{i+1} \in D_i, \text{ where } D_i \in \mathcal{D}, \ i = 0, 1, \ldots, t-1,$$

is called a *path* connecting the vertices x_0 and x_t [3]. The hypergraph $\mathcal{H} = (X, \mathcal{D})$ is called *connected* if for any pair of its vertices there is a path connecting them [3]. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called *circular* if there exists a host cycle on the vertex set X such that every C-edge and every D-edge induces a connected subgraph of the host cycle [2]. In other words, for a circular mixed hypergraph, there is always a circular ordering of the vertex set X, say $X = \{x_1, x_2, \ldots, x_n, x_1\}$, such that every edge induces an interval in this ordering [3]. For each $\ell \geq 2$, we denote $\mathcal{D}$ by $\mathcal{D}_\ell$ if and only if every ℓ consecutive vertices of X form a D-edge. Thus the mixed hypergraph $(X, \emptyset, \mathcal{D}_2)$ is a simple classical cycle on n vertices [2].

Every proper λ-coloring induces a partition of vertex set into color classes. Such partition, $\{X_1, X_2, \ldots, X_\lambda\}$, is called a *feasible partition* with respect to the coloring. The number of feasible partitions into λ colors is denoted by r_k. The vector $R(\mathcal{H}) = (r_1, \ldots, r_n)$ is called the *chromatic spectrum* of the mixed hypergraph $\mathcal{H}$ [3]. Each feasible partition into i color classes determines $i!$ strict i-colorings obtained from each other by a permutation of colors. In general, if we have $\lambda \geq i$ colors, then to count proper λ-colorings we have $\binom{\lambda}{i}$ ways to choose the subset of i colors. Therefore the number of proper λ-colorings generated by all feasible partitions into i subsets is $r_i\binom{\lambda}{i}i! = r_i\lambda(\lambda-1)\ldots(\lambda-i+1) = r_i\lambda^{(i)}$, which means $\binom{\lambda}{i}i!$ is denoted by $\lambda^{(i)}$ [3]. Let $\mathcal{P}(\mathcal{H}, \lambda)$ be a chromatic polynomial of a colorable mixed hypergraph $\mathcal{H}$ in λ and we have the following formula:

$$\mathcal{P}(\mathcal{H}, \lambda) = \sum_{i=\chi(\mathcal{H})}^{\bar{\chi}(\mathcal{H})} r_i(\mathcal{H})\lambda^{(i)}$$

Then we call $\mathcal{P}(\mathcal{H}, \lambda)$ the *chromatic polynomial* of the mixed hypergraph $\mathcal{H}$ [1].

In this thesis, we want to find $\mathcal{P}((X, \binom{X}{k}, D_2), \lambda)$, where $k = 4, 5, \ldots, n$, $4 \leq |X| \leq n$, and $k \leq n$. Here, $(X, \binom{X}{k}, D_2)$ is denoted by $\mathcal{H}_k^{(|X|)}$. Note that if $|X| = 3$, then the mixed hypergraph has no colorings, so we discuss the chromatic polynomial from $n = 4$. Also, when $k = 3$, the upper chromatic number is equal to two, so we don't focus on this result because it is trivial.

In Chapter 2, we use the splitting-contraction algorithm [1] which allows us to compute the chromatic polynomial. This idea is to find a pair of vertices that is neither a C-edge nor a D-edge, and to split all the colorings of $\mathcal{H}$ into two classes with respect to this pair of vertices [3]. Since it is complicated to calculate the number of two-coloring and three-colorng of the mixed hypergraph, we only need to check the first step of splitting-contraction algorithm. At the same time, we get a recurrence relation. According to this recurrence relation, we can solve it, and the solution is what we want.

In Chapter 3, we investigate the chromatic polynomial when $k \geq 4$. So all the chromatic polynomial is found out by the first step of splitting-contraction algorithm.

In Chapter 4, we solve the recurrence relation and get the answer for the chromatic polynomial. Also, we find out some relations between a summation form of $n!$ and our solution of the recurrence relation. Finally, we write a future study which associates with the mixed hypergraph which we discuss in this thesis.