Chapter 2

Some Obvious Cases

As we mentioned in the introduction, we will calculate the polynomial of the mixed hypergraph $(X, \binom{X}{k}, D_2)$, where $k = 4, 5, \ldots, n$, $4 \leq |X| \leq n$, and $k \leq n$. Here, we need splitting-contraction algorithm as follows:

Algorithm 2.1 $(splitting - contraction)$[1]

INPUT: *An arbitrary mixed hypergraph* $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *with* X *labeled* $1, 2, \ldots, n$.

OUTPUT: *A list* L *of all strict colorings, the chromatic spectrum* $\mathcal{R}(\mathcal{H})$*, the chromatic polynomial* $\mathcal{P}(\mathcal{H}, \lambda)$*, the chromatic numbers* $\chi(\mathcal{H})$ *and* $\overline{\chi}(\mathcal{H})$.

1. *Set lists* $L = Z = Y = \emptyset$, $\mathcal{R}(\mathcal{H}) = (0, 0, \ldots, 0)$, $\mathcal{P}(\mathcal{H}, \lambda) = 0$, $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = 0$. *Add* $\mathcal{H}$ *to* Y.
2. *Verify the condition of elimination for each element from Y; delete evidently uncolorable mixed hypergraphs from Y.*
3. *Perform* $\mathcal{C}$*-clearing and* $\mathcal{D}$*-clearing where possible in Y.*
4. *Perform contraction where possible in Y; when contracting, amalgamate the labels of the respective vertices.*
5. *Perform one splitting in each element of Y where possible; move complete* $\mathcal{D}$*-graphs from Y to Z; if splitting is performed at least once then go to step 2.*
6. *Form a list L of all strict colorings using the labels of vertices of complete* $\mathcal{D}$*-graphs from Z.*

7.Compute the chromatic spectrum $\mathcal{R}(\mathcal{H})$ by counting the numbers of complete $\mathcal{D}$-graphs in Z having exactly i vertices, $i = 1, 2, \ldots, n$.

8.Compute the chromatic polynomial $\mathcal{P}(\mathcal{H}, \lambda)$.

9.Determine $\chi(\mathcal{H})$, $\overline{\chi}(\mathcal{H})$ using $\mathcal{R}(\mathcal{H})$.

*10.**OUTPUT**: list L, vector $\mathcal{R}(\mathcal{H})$, polynomial $\mathcal{P}(\mathcal{H}, \lambda)$, numbers $\chi(\mathcal{H})$, $\overline{\chi}(\mathcal{H})$.*

End

In the next section, we start our task by this useful tool.

2.1 Find $\mathcal{P}(\mathcal{H}_4^{(n)}, \lambda)$.

We need some lemma to obtain the chromatic polynomial of $\mathcal{H}_4^{(n)}$.

Lemma 2.1 *If $n = 4$, $X = \{1, 2, 3, 4\}$, $\mathcal{C} = \{\binom{X}{4}\}$, and $\mathcal{D} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$, then $\mathcal{P}(\mathcal{H}_4^{(4)}, \lambda) = 1{\cdot}\lambda^{(2)} + 2{\cdot}\lambda^{(3)}$.*

Proof.

Clearly, since four vertices in X form a C-edge $\{1, 2, 3, 4\}$, we can not use four or more than four colors to color this mixed hypergraph $\mathcal{H}_4^{(4)}$. However, there is an unique way to color $\mathcal{H}_4^{(4)}$ when using two colors. That is $c(1) = c(3) = 1$ and $c(2) = c(4) = 2$ (or $\{1, 3\}\cup\{2, 4\}$).

Now, the remaining case is that using three colors color $\mathcal{H}_4^{(4)}$. We can easily get two ways to color it. That is (I) $c(1) = c(3) = 1$, $c(2) = 2$, and $c(4) = 3$ (or $\{1, 3\} \cup \{2\} \cup \{4\}$); (II) $c(1) = 1$, $c(2) = c(4) = 2$, and $c(3) = 3$ (or $\{1\} \cup \{2, 4\} \cup \{3\}$). So, we get $\mathcal{P}(\mathcal{H}_4^{(4)}, \lambda) = 1{\cdot}\lambda^{(2)} + 2{\cdot}\lambda^{(3)}$.

Also, by splitting-contraction algorithm and since $\mathcal{C}$-edge is $\binom{X}{4}$, we consider $\mathcal{D}$-graph only.

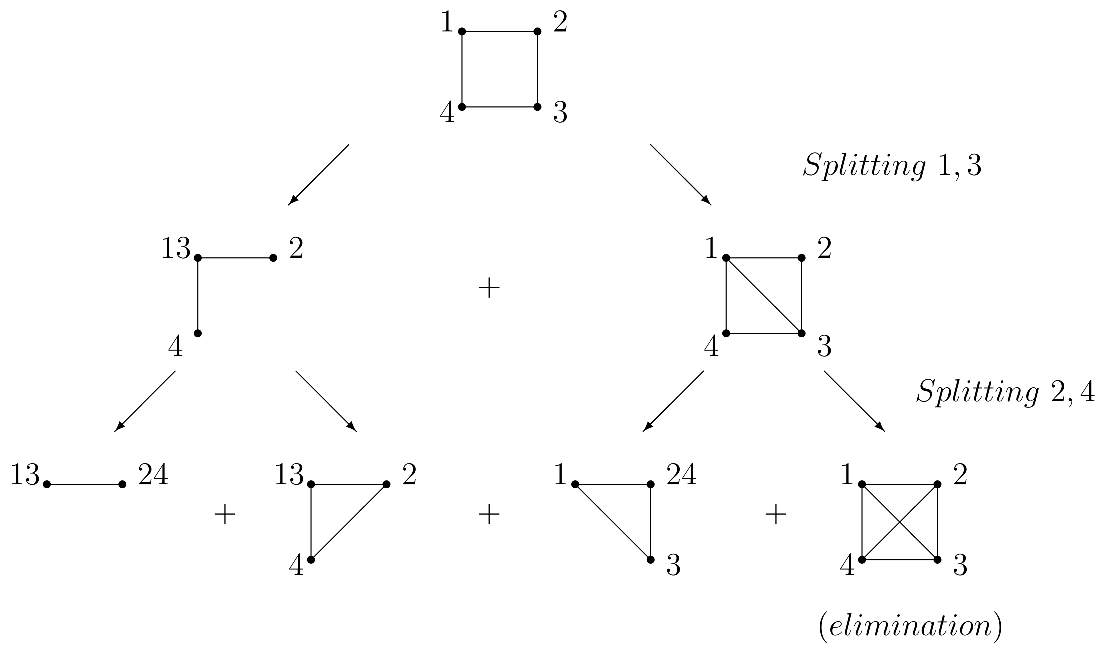

Figure 2.1

Figure 2.1 displays the colorings of the mixed hypergraph $\mathcal{H}_4^{(4)}$ and get the same chromatic polynomial.

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Thus, from Figure 2.1, we find out $\mathcal{P}(\mathcal{H}_4^{(4)}, \lambda) = 1{\cdot}\lambda^{(2)} + 2{\cdot}\lambda^{(3)}$, and now, we will see the mixed hypergraph with $n = 5$ for $k = 4$.

Lemma 2.2 *If* $n = 5$, $X = \{1, 2, 3, 4, 5\}$, $\mathcal{C} = \{\binom{X}{4}\}$, *and* $\mathcal{D} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$, *then* $\mathcal{P}(\mathcal{H}_4^{(5)}, \lambda) = 0{\cdot}\lambda^{(2)} + 5{\cdot}\lambda^{(3)}$.

Proof.

As above, we can not use four or more than four colors to color $\mathcal{H}_4^{(5)}$.

Trivially, since the $\mathcal{D}$-graph of $\mathcal{H}_4^{(5)}$ forms a cycle of five vertices, we can not use two colors to color $\mathcal{H}_4^{(5)}$. By splitting-contraction algorithm, there are five ways to color $\mathcal{H}_4^{(5)}$ when using three colors.

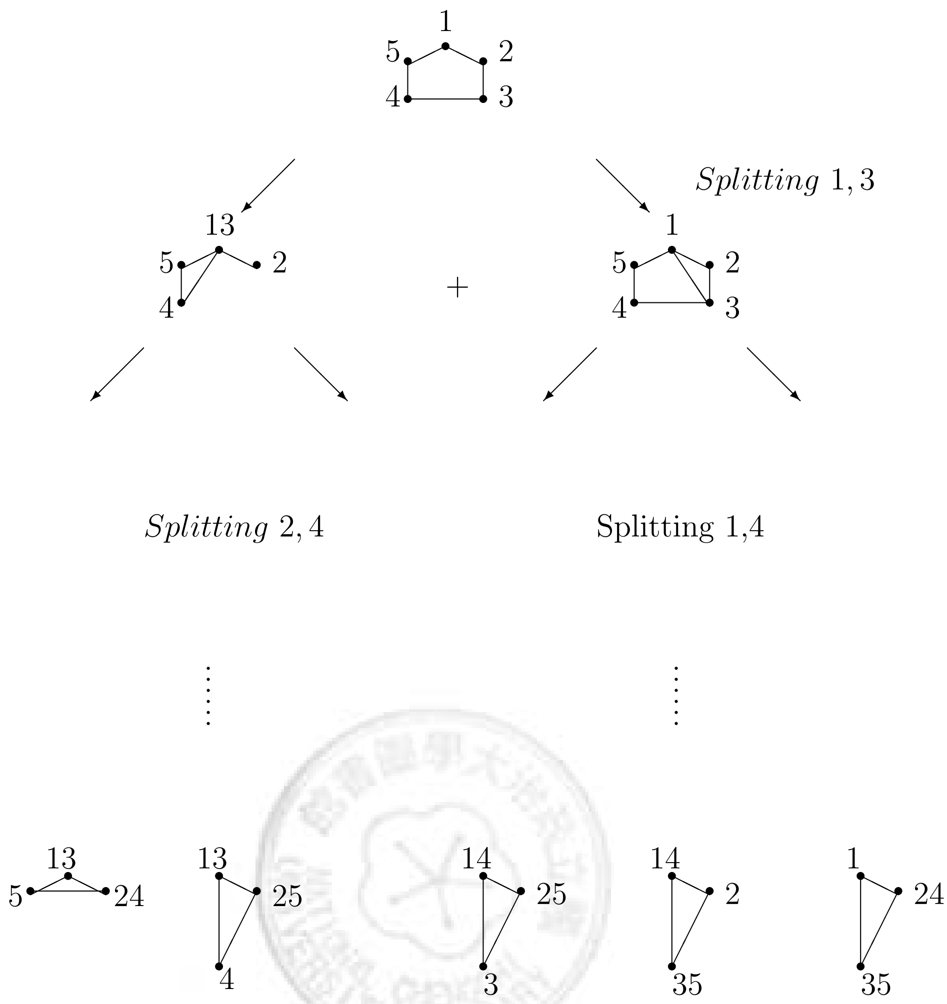

Figure 2.2

From Figure 2.2, we get five feasible partitions of colorings: (I) $\{1,3\}\cup\{2,4\}\cup\{5\}$, (II) $\{1,3\}\cup\{2,5\}\cup\{4\}$, (III) $\{1,4\}\cup\{2,5\}\cup\{3\}$, (IV) $\{1,4\}\cup\{2\}\cup\{3,5\}$, (V) $\{1\}\cup\{2,4\}\cup\{3,5\}$, and the others are eliminated.

So, we have shown that $\mathcal{P}(\mathcal{H}_4^{(5)}, \lambda) = 0\cdot\lambda^{(2)} + 5\cdot\lambda^{(3)}$.

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We have shown that the chromatic polynomial of the mixed hypergraph $\mathcal{H}_4^{(n)}$ when $n = 4$ and 5. Note that it is complicated if you analyze this mixed hypergraph to the last step of splitting-contraction algorithm. Also, we can't see the rule of the chromatic polynomial of $\mathcal{H}_4^{(n)}$ now. Hence, we discuss $n = 6$ and $n = 7$ for $\mathcal{H}_4^{(n)}$.

Lemma 2.3 *If* $n = 6$*,* $X = \{1, 2, 3, 4, 5, 6\}$*,* $\mathcal{C} = \{\binom{X}{4}\}$*, and* $\mathcal{D} = \{\{1, 2\}, \{2, 3\}, \{3, 4\},$ $\{4, 5\}, \{5, 6\}, \{6, 1\}\}$*, then* $\mathcal{P}(\mathcal{H}_4^{(6)}, \lambda) = 1{\cdot}\lambda^{(2)} + 10{\cdot}\lambda^{(3)}$.

Proof.

Similarly, we can not use four or more than four colors to color $\mathcal{H}_4^{(6)}$, and there is an unique way to color $\mathcal{H}_4^{(6)}$ when using two colors, say, $c(1) = c(3) = c(5) = 1$ and $c(2) = c(4) = c(6) = 2$.

If we use whole splitting-contraction algorithm, then there are so many steps to get the answer when n is large. In fact, when using spitting-contraction algorithm, we introduce an easy way to get the final answer by the first step of splitting and contraction. Here, we take this way when $n = 6$:

Since the $\mathcal{C}$-edges of $\mathcal{H}_4^{(6)}$ form a complete $\mathcal{C}$-hypergraph, the coloring we want is the remaining $\mathcal{D}$-hypergraph which is colored when using two or three colors only.

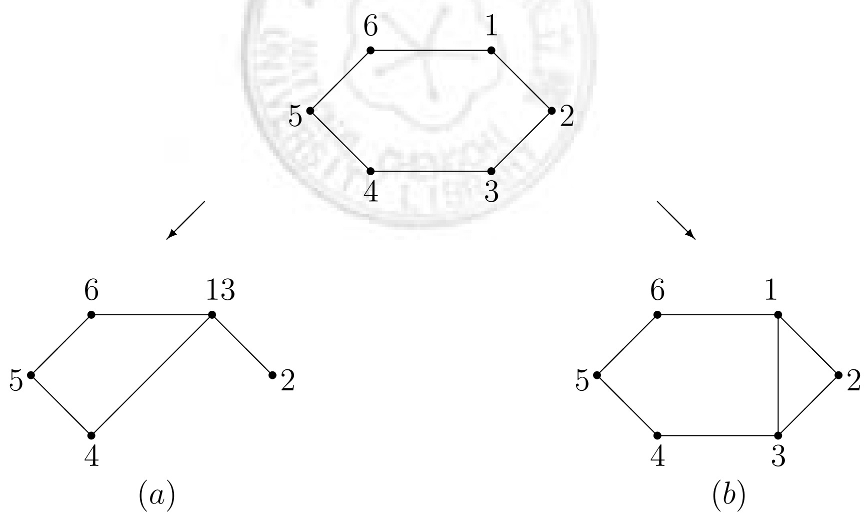

Figure 2.3

From (a) of Figure 2.3 , $X = \{13, 2, 4, 5, 6\}$, $\mathcal{C} = \{\{13, 2, 4, 5\}, \{13, 2, 4, 6\}, \{13, 2, 5, 6\},$ $\{13, 4, 5, 6\}, \{2, 4, 5, 6\}\}$, $\mathcal{D} = \{\{13, 2\}, \{13, 4\}, \{4, 5\}, \{5, 6\}, \{6, 13\}\}$. Since the $\mathcal{C}$-edges of (a) subject to the vertex-coloring, we only color the remaining $\mathcal{D}$-hypergraph. This time, we use three colors to color (a). However, we can consider the graph with the vertices

$\{13, 4, 5, 6\}$ which forms a smaller graph.

If these four vertices are colored with two colors, then "2" will be colored a new color other than "13". That is $c(13) = c(5) = 1$, $c(4) = c(6) = 2$, $c(2) = 3$.

If these four vertices are colored with three colors, then "2" will be colored one of these three colors other than "13". We have the following colorings:

$c(13) = c(5) = 1$, $c(4) = 2$, $c(6) = 3$, $c(2) = 2$,

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$c(13) = 1$, $c(4) = c(6) = 2$, $c(5) = 3$, $c(2) = 2$,

$c(13) = 1$, $c(4) = c(6) = 2$, $c(5) = 3$, $c(2) = 3$.

So (a) can be colored in $(1{\cdot}1) + (2{\cdot}2)$ ways.

Also, from (b) of Figure 2.3, $X = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{C} = \{\binom{X}{4}\}$, $\mathcal{D}$=$\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}, \{1, 3\}\}$. Since the $\mathcal{C}$-edges of (b) subject to the vertex-coloring, we can only try the remaining $\mathcal{D}$-hypergraph. As above, first we consider the graph with vertices $\{1, 3, 4, 5, 6\}$ which forms a smaller graph.

The induced subhypergraph with vertices $\{1, 3, 4, 5, 6\}$ can not be colored with two colors, so the remaining case is that if these five vertices are colored with three colors, then "2" will be colored one of these three colors other than "1" and "3". We have the following colorings:

$c(1) = c(4) = 1$, $c(3) = c(5) = 2$, $c(6) = 3$, $c(2) = 3$,

$c(1) = c(4) = 1$, $c(3) = c(6) = 2$, $c(5) = 3$, $c(2) = 3$,

$c(1) = c(5) = 1$, $c(3) = c(6) = 2$, $c(4) = 3$, $c(2) = 3$,

$c(1) = c(5) = 1$, $c(3) = 2$, $c(4) = c(6) = 3$, $c(2) = 3$,

$c(1) = 1$, $c(3) = c(5) = 2$, $c(4) = c(6) = 3$, $c(2) = 3$.

We can see that (b) can be colored in $(0 + 5{\cdot}1)$ ways.

In conclusion, the ways to color $\mathcal{H}_4^{(6)}$ with three colors are $[(1{\cdot}1)+(2{\cdot}2)]+[0+(5{\cdot}1)] = 10$. So, we get $\mathcal{P}(\mathcal{H}_4^{(6)}, \lambda) = 1{\cdot}\lambda^{(2)} + 10{\cdot}\lambda^{(3)}$.

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Lemma 2.4 *If* $n = 7$, $X = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{C} = \{\binom{X}{4}\}$, *and* $\mathcal{D} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 1\}\}$, *then* $\mathcal{P}(\mathcal{H}_4^{(7)}, \lambda) = 0{\cdot}\lambda^{(2)} + 21{\cdot}\lambda^{(3)}$.

Proof.

Since n is odd, the hypergraph is never colored with two colors, we consider the colorings with three colors.

(c) (d)

Figure 2.4

As above, we use the first step of splitting-contraction algorithm, and get part (c) and part (d).

From part (c) of Figure 2.4, similarly as Lemma 2.3, the five vertices $\{13, 4, 5, 6, 7\}$ are colored with three colors in five ways and "2" can be colored one of these three colors other than "13". So (c) can be colored in (5·2) ways.

From part (d) of Figure 2.4, similarly as Lemma 2.3, the six vertices $\{1, 3, 4, 5, 6, 7\}$ are colored with two colors in one way and "2" can be colored the third color in one way. If the six vertices are colored with three colors, then "2" will be colored other than "1" and "3".

In conclusion, the ways to color $\mathcal{H}_4^{(7)}$ with three colors are $(5\cdot2)+[(1\cdot1)+(10\cdot1)] = 21$. So, we obtain $\mathcal{P}(\mathcal{H}_4^{(7)}, \lambda) = 0\cdot\lambda^{(2)} + 21\cdot\lambda^{(3)}$.

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As previous lemmas, we have shown that the simplest case of the mixed hypergraph $\mathcal{H}_4^{(n)}$, and then we will prove the general case for $\mathcal{H}_4^{(n)}$.

Theorem 2.1 *Show that* $\mathcal{P}(\mathcal{H}_4^{(n)}, \lambda) = \chi_n^{(2)} \cdot \lambda^{(2)} + \Pi_n^{(3)} \cdot \lambda^{(3)}$*,* $n \geq 4$ *is an integer, where*

$$\chi_n^{(2)} = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases},$$

$$\Pi_n^{(3)} = \Pi_{n-1}^{(3)} + 2\Pi_{n-2}^{(3)} + 1.$$

Proof.

We will claim that

$$\Pi_n^{(3)} = \begin{cases} (\Pi_{n-2}^{(3)} \cdot 2) + (1 + \Pi_{n-1}^{(3)}) & \text{if n is odd} \\ (1 + \Pi_{n-2}^{(3)} \cdot 2) + (\Pi_{n-1}^{(3)}) & \text{if n is even.} \end{cases}$$

We know that $\Pi_4^{(3)} = 2$ and $\Pi_5^{(3)} = 5$. We prove the theorem by induction on n. If $n = 4$, it is true by Lemma 2.1, and if $n = 5$, it is also true by Lemma 2.2. Now, we let $n \leq k$ hold and we will show case by case.

If $n = k+1$ is even, then k is odd and $k-1$ is even. Clearly, $k+1$ is even implies there is one of feasible partitions of the mixed hypergraph $\mathcal{H}_4^{(k+1)}$ into two sets. That is $\chi_n^{(2)}$ =1, if $n = k+1$ is even. Also, because the $\mathcal{C}$-edges are $\{\binom{X}{4}\}$, there are none of feasible partitions of the mixed hypergraph $\mathcal{H}_4^{(k+1)}$ into four sets. i.e. $\mathcal{P}(\mathcal{H}_4^{(n)}, \lambda) = 0 \cdot \lambda^{(2)} + b \cdot \lambda^{(3)}$ for some $b \in Z$. The remaining case is to find "b". Since $n = k+1$ is even, by the first step of splitting-contraction algorithm, we obtain two mixed hypergraphs.

One of which is contracting the vertices "1" and "3", and the $\mathcal{C}$-edges of the hypergraph subject to the vertex-coloring, all of the $\mathcal{C}$-edges can be deleted, because the answer we want is automatically true for the $\mathcal{D}$-hypergraph colorings. Now, we consider the $\mathcal{D}$-hypergraph with the vertices $\{13, 4, 5, 6, ..., k+1\}$, it forms a smaller case ($n = k-1$) colored with two or three colors. When using two colors, there is one way to color this graph and "2" will be colored with the coloring other than "13" in one way. Therefore, there is (1·1) way to color this graph. When using three colors, there is $\Pi_{k-1}^{(3)}$ ways to color this graph and "2" will be colored with the coloring other than "13" in two ways. That is to say that there are $\Pi_{k-1}^{(3)} \cdot 2$ ways to color this graph.

On the other hand, there is a mixed hypergraph connecting "1" and "3", and the $\mathcal{C}$-edges still can be deleted, because the answer we want is automatically true for the $\mathcal{D}$-hypergraph colorings. Now, as above, consider the $\mathcal{D}$-hypergraph with the vertices

$\{1, 3, 4, 5, 6, ..., k+1\}$, it forms a smaller case ($n = k$) colored with only three colors. There are $\Pi_k^{(3)}$ ways to color this graph and "2" will be colored with the coloring other than "1" and "3" in one way. Hence, there are $\Pi_k^{(3)} \cdot 1$ ways to color this graph. But there are none of feasible partition of the mixed hypergraph with n vertices because $n = k$ is odd. Finally, we get $[(1 \cdot 1) + \Pi_{k-1}^{(3)} \cdot 2] + (\Pi_k^{(3)} \cdot 1)$ ways to color $\mathcal{H}_4^{(k+1)}$ with three colors when $k+1$ is even.

Similarly, if $n = k+1$ is odd, then k is even and $k-1$ is odd. Obviously, $k+1$ is odd implies there are none of feasible partitions of the mixed hypergraph $\mathcal{H}_4^{(k+1)}$ into two sets. That is $\chi_n^{(2)} = 0$, if $n = k+1$ is odd. Also, because the $\mathcal{C}$-edges are $\{\binom{X}{4}\}$, there are none of feasible partitions of the mixed hypergraph $\mathcal{H}_4^{(k+1)}$ into four sets. Hence, $\mathcal{P}(\mathcal{H}_4^{(n)}, \lambda) = 0 \cdot \lambda^{(2)} + d \cdot \lambda^{(3)}$ for some $d \in Z$. The remaining case is to find "d". Since $n = k+1$ is odd, by the first step of splitting-contraction algorithm as above, we also obtain two mixed hypergraphs.

One of the hypergraph is contracting the vertices "1" and "3", which means "1" and "3" will be paint by the same color, and the $\mathcal{C}$-edges of the hypergraph still subject to the vertex-coloring, all of the $\mathcal{C}$-edges can be ignored, because the answer we want is automatically true for the $\mathcal{D}$-hypergraph colorings. Now, we consider the $\mathcal{D}$-hypergraph with the vertices $\{13, 4, 5, 6, ..., k+1\}$, it forms a smaller case ($n = k-1$) colored with two or three colors. When using two colors, there are no ways to color this graph because $n = k-1$ is odd. When using three colors, there is $\Pi_{k-1}^{(3)}$ ways to color this graph and "2" will be colored with the coloring other than "13" in two ways, which means there are $\Pi_{k-1}^{(3)} \cdot 2$ ways to color this graph.

On the other hand, there is a mixed hypergraph splitting "1" and "3", then we add a D-edge between "1" and "3" by splitting-contraction algorithm. However, the $\mathcal{C}$-edges still can be ignored, because the answer we want is automatically true for the $\mathcal{D}$-hypergraph colorings. As above, consider the $\mathcal{D}$-hypergraph with the vertices $\{1, 3, 4, 5, 6, ..., k+1\}$, it forms a smaller case ($n = k$) colored with two or three colors. If we use two colors, the $\mathcal{D}$-hypergraph has only one way, so "2" will be colored in one way. Obviously, there are $\Pi_k^{(3)}$ ways to color this graph and "2" will be colored with the coloring other than "1" and "3" in one way. Therefore, there are $[(1 \cdot 1) + (\Pi_k^{(3)} \cdot 1)]$ ways to color this graph with splitting "1" and "3". Finally, we get $[\Pi_{k-1}^{(3)} \cdot 2] + [(1 \cdot 1) + (\Pi_k^{(3)} \cdot 1)]$ ways to color $\mathcal{H}_4^{(k+1)}$

with three colors when $k+1$ is odd.

Now, we have proved that

$$\Pi_n^{(3)} = \begin{cases} (\Pi_{n-2}^{(3)} \cdot 2) + (1 + \Pi_{n-1}^{(3)}) & \text{if n is odd} \\ (1 + \Pi_{n-2}^{(3)} \cdot 2) + (\Pi_{n-1}^{(3)}) & \text{if n is even.} \end{cases}$$

which means $\Pi_n^{(3)} = \Pi_{n-1}^{(3)} + 2\Pi_{n-2}^{(3)} + 1$ whenever n is odd or even.

Since $\Pi_n^{(3)} = \Pi_{n-1}^{(3)} + 2\Pi_{n-2}^{(3)} + 1$ and $\Pi_4^{(3)} = 2$ and $\Pi_5^{(3)} = 5$, by solving the recurrence relation, we get:

$$\Pi_n^{(3)} = \frac{1}{6}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}.$$

❑

In this chatper, we have displayed the chromatic polynomial of the mixed hypergraph $\mathcal{H}_4^{(n)}$, and got the following corollary.

Corollary 2.1 *From Theorem 2.1,* $\mathcal{P}(\mathcal{H}_4^{(n)}, \lambda) = (\chi_n^{(2)})\lambda^{(2)} + (\Pi_n^{(3)})\lambda^{(3)}$, *where*

$$\chi_n^{(2)} = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases},$$

$$\Pi_n^{(3)} = \frac{1}{6}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}.$$