Chapter 3

The Relations of $\mathcal{P}(\mathcal{H}_k^{(n)}, \lambda)$

3.1 Find $\mathcal{P}(\mathcal{H}_5^{(n)}, \lambda)$

From previous chapter, we introduce a way to search $\mathcal{P}(\mathcal{H}_4^{(n)}, \lambda)$, now, let us find $\mathcal{P}(\mathcal{H}_5^{(n)}, \lambda)$.

We know that the $\mathcal{C}$-edges of $\mathcal{H}_5^{(n)}$ form a complete $\mathcal{C}$-hypergraph, so the colorings must be with two, three, or four colors. First, when using two colors to color $\mathcal{H}_5^{(n)}$, there is one of feasible partitions of the mixed hypergraph $\mathcal{H}_5^{(n)}$ into two sets if the number of vertices is even; obviously, there are none of feasible partitions of the mixed hypergraph $\mathcal{H}_5^{(n)}$ into two sets if the number of vertices is odd. This case is the same as $\mathcal{H}_4^{(n)}$. Second, by the first step of splitting-contraction algorithm, we want to find the recurrence relation for $\lambda^{(3)}$. As before, the feasible partitions of the mixed hypergraph $\mathcal{H}_5^{(n)}$ into three sets are $\Pi_n^{(3)}$, which is shown in Theorem 2.1. Third, the remaining work is to find the recurrence relation for $\lambda^{(4)}$ of $\mathcal{H}_5^{(n)}$.

Also, as above, by the first step of splitting-contraction algorithm, we get two part of the mixed hypergraph. The mixed hypergraph with contracting vertices "1" and "3" has a smaller mixed hypergraph with vertices $\{13, 4, 5, \ldots, n\}$ which is the smaller case with $(n-2)$ vertices. The $\mathcal{C}$-edges of which subject to the upper chromatic number of colorings, so we focus on the $\mathcal{D}$-hypergraph (or $\mathcal{D}$-graph). When using three colors, the ways to color the graph are $\Pi_{n-2}^{(3)}$, then "2" will be colored with the fourth color. When using four colors, the ways to color the graph are $\Pi_{n-2}^{(4)}$, then "2" will be colored with one of the four colors other than "13". The mixed hypergraph with connecting vertices "1" and "3" has a smaller mixed hypergraph with vertices $\{1, 3, 4, \ldots, n\}$ which is the smaller case with $(n-1)$ vertices. When using three colors, the ways to color the graph are $\Pi_{n-1}^{(3)}$, then "2" will be colored with the fourth color. When using four colors, the ways to color the graph are $\Pi_{n-1}^{(4)}$, then "2" will be colored with one of these four colors other than "1" and "3".

So, we obtain $\Pi_n^{(4)} = (\Pi_{n-2}^{(3)} \cdot 1 + \Pi_{n-2}^{(4)} \cdot 3) + (\Pi_{n-1}^{(3)} \cdot 1 + \Pi_{n-1}^{(4)} \cdot 2)$

$$\Longleftrightarrow \Pi_n^{(4)} = 2\Pi_{n-1}^{(4)} + 3\Pi_{n-2}^{(4)} + (\Pi_{n-1}^{(3)} + \Pi_{n-2}^{(3)})$$
$$\Longleftrightarrow \Pi_n^{(4)} = 2\Pi_{n-1}^{(4)} + 3\Pi_{n-2}^{(4)} + (\frac{1}{2}2^{n-2} - 1), \text{ where } \Pi_4^{(4)} = 1, \text{ and } \Pi_3^{(4)} = 0$$
$$\Longleftrightarrow \Pi_n^{(4)} = \frac{1}{24}3^n + \frac{1}{24}(-1)^n - \frac{1}{6}2^n + \frac{1}{4}.$$

Lemma 3.1 *$\mathcal{P}(\mathcal{H}_5^{(n)}, \lambda) = (\chi_n^{(2)})\lambda^{(2)} + (\Pi_n^{(3)})\lambda^{(3)} + (\Pi_n^{(4)})\lambda^{(4)}$, where*

$$\chi_n^{(2)} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases},$$
$$\Pi_n^{(3)} = \frac{1}{6}2^n - \frac{1}{6}(-1)^n - \frac{1}{2},$$
$$\text{and } \Pi_n^{(4)} = \frac{1}{24}3^n + \frac{1}{24}(-1)^n - \frac{1}{6}2^n + \frac{1}{4}.$$

3.2 Find $\mathcal{P}(\mathcal{H}_k^{(n)}, \lambda)$

Similarly, $\mathcal{P}(\mathcal{H}_6^{(n)}, \lambda)$ must be $(\chi_n^{(2)})\lambda^{(2)} + (\Pi_n^{(3)})\lambda^{(3)} + (\Pi_n^{(4)})\lambda^{(4)} + (\Pi_n^{(5)})\lambda^{(5)}$.

From now on, we have to search $\Pi_n^{(5)}$. By the first step of splitting-contraction algorithm, $\Pi_n^{(5)} = (\Pi_{n-2}^{(4)} \cdot 1 + \Pi_{n-2}^{(5)} \cdot 4) + (\Pi_{n-1}^{(4)} \cdot 1 + \Pi_{n-1}^{(5)} \cdot 3)$

$$\Longleftrightarrow \Pi_n^{(5)} = 3\Pi_{n-1}^{(5)} + 4\Pi_{n-2}^{(5)} + (\Pi_{n-1}^{(4)} + \Pi_{n-2}^{(4)}), \text{ where } \Pi_5^{(5)} = 1, \text{ and } \Pi_4^{(5)} = 0$$
$$\Longleftrightarrow \Pi_n^{(5)} = \frac{1}{120}4^n - \frac{1}{120}(-1)^n - \frac{1}{24}3^n + \frac{1}{12}2^n - \frac{1}{12}.$$

Lemma 3.2 $\mathcal{P}(\mathcal{H}_6^{(n)}, \lambda) = (\chi_n^{(2)})\lambda^{(2)} \ + \ (\Pi_n^{(3)})\lambda^{(3)} \ + \ (\Pi_n^{(4)})\lambda^{(4)} \ + \ (\Pi_n^{(5)})\lambda^{(5)}$, *where*

$$\chi_n^{(2)} = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases},$$

$$\Pi_n^{(3)} = \frac{1}{6}2^n - \frac{1}{6}(-1)^n - \frac{1}{2},$$

$$\Pi_n^{(4)} = \frac{1}{24}3^n + \frac{1}{24}(-1)^n - \frac{1}{6}2^n + \frac{1}{4},$$

$$\Pi_n^{(5)} = \frac{1}{120}4^n - \frac{1}{120}(-1)^n - \frac{1}{24}3^n + \frac{1}{12}2^n - \frac{1}{12}.$$

We know that the $\mathcal{C}$-edges of the mixed hypergraph $\mathcal{H}_k^{(n)}$ form a k-uniform $\mathcal{C}$-hypergraph, so the colorings of $\mathcal{H}_k^{(n)}$ are neither greater than nor equal to k.

Lemma 3.3 *When the* $\mathcal{C}$*-edges are* $\binom{n}{k}$, *we have*

$$\mathcal{P}(\mathcal{H}_k^{(n)}, \lambda) = (\chi_n^{(2)})\lambda^{(2)} + \sum_{i=3}^{k-1}(\Pi_n^{(i)})\lambda^{(i)}, \ \textit{where } k = 4, 5, \ldots, n,$$

$$\textit{and } \Pi_n^{(k)} = [(k-1)\Pi_{n-2}^{(k)} + \Pi_{n-2}^{(k-1)}] + [(k-2)\Pi_{n-1}^{(k)} + \Pi_{n-1}^{(k-1)}].$$

The remaining work we have to discuss in Chapter 4 is how to calculate $\Pi_n^{(i)}$, where $i = 3, 4, \ldots, n-1$.