Chapter 4

Solving $\Pi_n^{(k)}$ for $\lambda^{(k)}$

4.1 Solutions for $\Pi_n^{(k)}$

The following theorem is the solution of the recurrence relation which we discuss in the previous chapter.

Theorem 4.1 *The solution of the recurrence relation*

$$\Pi_n^{(k)} = (k-1)\Pi_{n-2}^{(k)} + (k-2)\Pi_{n-1}^{(k)} + (\Pi_{n-2}^{(k-1)} + \Pi_{n-1}^{(k-1)}) \textit{ with}$$

$$\Pi_n^{(2)} = \chi_n^{(2)} = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases},$$

$$\Pi_n^{(3)} = \frac{1}{6}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}$$

$$\textit{is} \quad \Pi_n^{(k)} = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-1-i)^n, \textit{ where } k = 3, 4, \ldots, n-1.$$

Proof.

The recurrence relation has general solution and particular solution. The roots of the general solution of $\Pi_n^{(k)}$ is $k-1$ and -1 by solving $\Pi_n^{(k)} = (k-1)\Pi_{n-2}^{(k)} + (k-2)\Pi_{n-1}^{(k)}$. The particular solution of $\Pi_n^{(k)}$ is obtained from $\Pi_{n-2}^{(k-1)}$ and $\Pi_{n-1}^{(k-1)}$ which have their solutions, and by Theorem 2.1, we have $\Pi_n^{(3)} = \Pi_{n-1}^{(3)} + 2\Pi_{n-2}^{(3)} + 1$, so the particular roots are $k-2$,

$k-3, \ldots, 2, 1$. Obviously, the roots of the recurrence relation are $k-1$, $k-2$, $k-3$, $\ldots$, 2, 1, and -1. Since the root 0 is always true in the equation, without loss of generality, we can add root 0 to the formula.

Now, we put $\Pi_n^{(k)} = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-1-i)^n$ into

$$\Pi_n^{(k)} = (k-1)\Pi_{n-2}^{(k)} + (k-2)\Pi_{n-1}^{(k)} + (\Pi_{n-2}^{(k-1)} + \Pi_{n-1}^{(k-1)}),$$

and we get $\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-1-i)^n$

$$= (k-1)\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-1-i)^{n-2} + (k-2)\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-1-i)^{n-1}$$
$$+\frac{1}{(k-1)!}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(k-2-i)^{n-2} + \frac{1}{(k-1)!}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(k-2-i)^{n-1}.$$

It suffices to show that $\sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-1-i)^n$

$$= (k-1)\sum_{i=0}^{k-1}\binom{k}{i}(-1)^i(k-1-i)^{n-2} + (k-2)\sum_{i=0}^{k}\binom{k}{i}(-1)^i(k-1-i)^{n-1}$$
$$+(k)\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(k-2-i)^{n-2} + (k)\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i(k-2-i)^{n-1} \text{..........(*)}.$$

We only need to consider the coefficient of each root from right hand side (RHS) and left hand side (LHS) of (*).

(1) As above, the coefficient of 0^n's has done.

(2) Consider the coefficient of 1^n's,

$$\text{LHS} = \binom{k}{k-2}(-1)^{n-4}$$
$$= \binom{k}{2}(-1)^n, \text{ and}$$

$$\text{RHS} = (k-1)\binom{k}{k-2}(-1)^{n-4} + (k-2)\binom{k}{k-2}(-1)^{n-4}$$

$$+\ (k)\binom{k-1}{k-3}(-1)^{n-5}+(k)\binom{k-1}{k-3}(-1)^{n-5}$$

$$=(-1)^n\frac{1}{2}(k-1)(k)[(k-1)+(k-2)-(k-2)-(k-2)]$$

$$=(-1)^n\binom{k}{2},$$

so LHS = RHS.

(3) For any root r, $r = 2, 3, \ldots, k-2$,

$$\text{LHS} = \binom{k}{k-(r+1)}(-1)^{k-r+1}(r)^n$$

$$= \binom{k}{r+1}(-1)^{k-r+1}(r)^n, \text{ and}$$

$$\text{RHS} = (k{-}1)\binom{k}{(k)-(r+1)}(-1)^{k-r+1}(r)^{n-2}+(k{-}2)\binom{k}{(k)-(r+1)}(-1)^{k-r+1}(r)^{n-1}$$

$$+(k)\binom{k-1}{(k-1)-(r+1)}(-1)^{k-r+2}(r)^{n-2}+(k)\binom{k-1}{(k-1)-(r+1)}(-1)^{k-r+2}(r)^{n-1}$$

$$=\binom{k}{r+1}(-1)^{k-r+1}[k-1+r(k-2)-(k-1-r)-r(k-1-r)](r)^{n-2}$$

$$=\binom{k}{r+1}(-1)^{k-r+1}(r)^n,$$

so LHS = RHS.

(4) For the root $k-1$,

$$\text{LHS} = \binom{k}{0}(-1)^2(k-1)^n$$

$$= (k-1)^n, \text{ and}$$

$$\text{RHS} = (k-1)\binom{k}{0}(-1)^2(k-1)^{n-2} + (k-2)\binom{k}{0}(-1)^2(k-1)^{n-1}$$

$$= (k-1)^{n-2}[k-1+(k-2)(k-1)]$$

$$= (k-1)^n,$$

so LHS = RHS.

(5) For the root -1,

$$\text{LHS} = \binom{k}{k}(-1)^{k+2}(-1)^n$$

$$= (-1)^{k+2}(-1)^n, \text{ and}$$

$$\text{RHS} = (k-1)\binom{k}{k}(-1)^{k+2}(-1)^{n-2} \ + \ (k-2)\binom{k}{k}(-1)^{k+2}(-1)^{n-1}$$

$$+ \ (k)\binom{k-1}{k-1}(-1)^{k+1}(-1)^{n-2} \ + \ (k)\binom{k-1}{k-1}(-1)^{k+1}(-1)^{n-1}$$

$$= (-1)^{k+2}(-1)^{n-2}[k-1+(k-2)(-1)] \ + \ 0$$

$$= (-1)^{k+2}(-1)^{n-2}$$

$$= (-1)^{k+2}(-1)^n,$$

so LHS = RHS.

Since LHS = RHS for all the root, we complete the proof.

❑

Corollary 4.1 *According to our symbol, we get* $\Pi_n^{(n)} = 1$*, which means complete* $\mathcal{D}$*-graph with* n *vertices has only one coloring when we use* n *colors to paint the vertices. That is*

$$n! = \sum_{i=0}^{n}\binom{n}{i}(-1)^i(n-1-i)^n.$$

Note that this complete $\mathcal{D}$-graph is eliminated by splitting-contraction algorithm.

Remark 4.1 *As everyone knows, $n!$ can be expressed by inclusion-exclusion formula:*

$$n! = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (n-i)^n,$$

which is similar to our Corollary 4.1.

Throughout this thesis, we get the conclusion as follows:

Corollary 4.2 *From Theorem 4.1, the chromatic polynomial of the mixed hypergraph $(X, \binom{X}{k}, D_2)$ is*

$$\mathcal{P}(\mathcal{H}_k^{(n)}, \lambda) = (\chi_n^{(2)})\lambda^{(2)} + \sum_{i=3}^{k-1} (\Pi_n^{(i)})\lambda^{(i)}, \text{ where } k = 4, 5, \ldots, n,$$

$$\chi_n^{(2)} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases},$$

$$\text{and } \Pi_n^{(k)} = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i (k-1-i)^n.$$

4.2 Future study

The main inspiration of this thesis comes from Section 9.6 of "*Coloring Mixed Hypergraphs: Theory, Algorithms and Applications*" ([3]). The mixed hypergraph $\mathcal{H}$ with complete $\mathcal{C}$-edges and $\mathcal{D}_2$ has finished in this thesis. Here is a future study for the reader.

Problem 4.1 *Is there a formula for the chromatic polynomial of an (ℓ, m)-uniform circular mixed hypergraph $\mathcal{H} = (X, \mathcal{C}_\ell, \mathcal{D}_m)$, $|X| \geq 4$, $\ell \geq 4$, $m \geq 3$?*