II. Literature Review

2.1. Valuation models for credit default swaps

Hull and White (2000) assume the only reason that a corporate bond worth less than a comparable Treasury bond is probability of default. The price difference between the two bonds is cost of the default by the company. Based on this concept, the risk-neutral probabilities of the default can be stripped out with a set of corporate bonds issued by the reference company (or by a company having the same risk of default with the reference company). The premium leg and the default leg of a CDS contract then can be calculated through this set of default probabilities. The fair price of a CDS contract (or CDS spread) thus is the rate of a payment that makes the premium leg equal to the default leg.

For example, suppose the prices of N corporate bonds with maturities $t_1 < t_2 \ldots < t_N$ are $B_1, B_2, \ldots B_N$, and the prices of the Treasury bonds with the same cash flows with the corporate bonds are $G_1, G_2 \ldots G_N$. For the j^{th} bond, we can write

$$B_j - G_j = \text{the present value of expected loss from default}$$

The present value of expected loss from default can be obtained from multiplying the loss from default by the default probability before the j^{th} bond matures. If the default probability between t_{i-1} and t_i is denoted as p_i , then the above formula can be represented as

$$B_j - G_j = \sum_{i=1}^{j} p_i \beta_{ij} \qquad (2.1)$$

where β_{ij} is the loss from default of the j^{th} bond between t_{i-1} and t_i.

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Rearranging Equation (2.1), the default probability p_j between t_{j-1} and t_j is

$$p_j = \frac{B_j - G_j - \sum_{i=1}^{j-1} p_i \beta_{ij}}{\beta_{jj}}. \qquad (2.2)$$

Divided by the respective time length, each default probability p_j produces the corresponding default density q_j.[1] Thus the survival probability at a certain time is

$$\pi(t) = 1 - \int_0^t q(u)du.$$

Suppose a forward CDS contract starts at time T_0 ,and the payment dates are at T_1, $T_2 \ldots T_n$. The spread for this contract can be expressed as

$$S_{0,n}(0) = \frac{\int_0^{T_n} (1-R)D(0,u)q(u)du}{\sum_{i=1}^{n} D(0,T_i)(T_i - T_{i-1}) \int_0^{T_n} q(u)du + \int_0^{T_n} A(u)D(0,u)q(u)du}, \qquad (2.3)$$

where R is recovery rate, $D(0,t)$ is the discount factor for a period from time zero to t, and $A(t)$ is accrual payment by protection buyer when default occurs at t.

[1] The default density is actually the same as the default intensity (or hazard rate). The relationship is $q(t) = h(t)e^{-\int_0^{\tau} h(\tau)d\tau}$. For more details, please see Hull and White (2000).

Brigo (2006) demonstrates that the discounted payoff of a forward CDS contract identical to the above example can be expressed as

$$\Pi(0) = D(0,\tau) \cdot 1_{(T_0<\tau\leq T_n)} \cdot (\tau - T_\tau) \cdot S_{0,n} + \sum_{i=1}^{n} D(t,T_i) \cdot (T_i - T_{i-1}) \cdot S_{0,n} \cdot 1_{(\tau\geq T_i)}$$

$$-D(t,\tau) \cdot 1_{(T_0<\tau\leq T_n)} \cdot (1-R), \tag{2.4}$$

where τ is the default time, and T_τ is the last payment date before default.

For easier algebra, Brigo makes some slight modifications to Equation (2.4). The accrual payment by the protection buyer is eliminated, and the protection payment by the protection seller is deferred to the first payment date that follows default. This type of the CDS contract is called as a postponed CDS contract, and the discounted payoff reduces to

$$\Pi_P(0) = \sum_{i=1}^{n} D(t,T_i) \cdot (T_i - T_{i-1}) \cdot S_{0,n} \cdot 1_{(\tau\geq T_i)} - \sum_{i=1}^{n} D(t,T_i) \cdot 1_{(T_{i-1}<\tau\leq T_i)} \cdot (1-R). \tag{2.5}$$

With the definition of postponed CDS contract, under the risk-neutral measure Q the forward CDS price can be computed as

$$CDS(0) = \sum_{i=1}^{n} (T_i - T_{i-1}) \cdot S_{0,n} \cdot E_Q\left[D(0,T_i)1_{(\tau\geq T_i)} \middle| G_0\right]$$

$$-\sum_{i=1}^{n} E_Q\left[D(0,T_i)1_{(T_{i-1}<\tau\leq T_i)} \middle| G_0\right] \cdot (1-R)$$

$$= \sum_{i=1}^{n} (T_i - T_{i-1}) \cdot S_{0,n} \cdot \bar{P}(0, T_i) - \sum_{i=1}^{n} E_Q\left[D(0, T_i) 1_{(T_{i-1} < \tau \leq T_i)} | G_0\right] \cdot (1 - R)$$

(2.6)

where G_t represents the filtration at time *t*, and $\bar{P}(t, T_i)$ is the corporate zero coupon bond.

The forward CDS spread denoted by *S* here is the spread that makes the value of a CDS contract zero.

$$S_{0,n}(0) = \frac{\sum_{i=1}^{n} E_Q\left[D(0, T_i) 1_{\{T_{i-1} < \tau \leq T_i\}} | G_0\right] \cdot (1 - R)}{\sum_{i=1}^{n} (T_i - T_{i-1}) \cdot \bar{P}(0, T_i)}. \quad (2.7)$$

2.2. Valuation models for a European CDS option

2.2.1. Hull and White (2002)

The primary concept of a forward CDS spread is the spread that makes the value of a CDS contract equal to zero. In other words, we can express the forward CDS spread as

$$S_{0,n} = \frac{DL}{PL}, \quad (2.8)$$

where *PL* means the present value of cash flows paid by protection buyer when the fixed payment is \$1, and *DL* represents the present value of cash flows paid by the protection seller when default occurs.

Hull and White (2002) use another formula to express the forward CDS spread.

$$S_{0,n} = \frac{DL^*}{PL^*} \ , \tag{2.9}$$

The difference between Equation (2.8) and Equation (2.9) is that DL^* and PL^* are conditional on no default prior to time T_0.

Under the numeraire PL^*, there is a probability measure so that all asset prices form martingales. Hull and White assume the forward CDS spread $S(t)$ is a lognormal distribution conditional on no default prior time T_0. Thus the dynamics of the CDS spread can be represented as

$$\frac{dS_{0,n}(t)}{S_{0,n}(t)} = \sigma \cdot dW^*(t), \tag{2.10}$$

where W^* is the Brownian motion under the respective measure denoted as Q^*.

Define $V(t)$ as the time t value of a European CDS option with exercise price K on a CDS contract starting from time T_0 to T_n. Under the probability measure Q^*, $V/_{PL^*}$ is also a martingale, the option value conditional on no default prior to time T can be expressed as

$$V^*(0) = PL^*(0) \cdot E_{Q^*}\left[V^*(T_0) /_{PL^*(T_0)} \,|G_0 \right]. \tag{2.11}$$

At T_0, $V^*(T_0) = PL^*(T_0) max(S(T_0) - K, 0)$. Thus (2.11) can be described as

$$V^*(0) = PL^*(0) \cdot E_{Q^*}\left[max\left(S_{0,n}(T_0) - K, 0\right) | G_0 \right]. \tag{2.12}$$

According to Black formula (1976), the option value conditional on no default prior T_0 can be further given by

$$V^*(0) = PL^*(0) \cdot \left[S_{0,n}(0)N(d_1) - KN(d_2)\right], \quad (2.13)$$

where $d_1 = \frac{1}{\sigma\sqrt{T_0}}\left(ln\left(\frac{S_{0,n}(0)}{K}\right) + \frac{1}{2}\sigma^2 \cdot T_0\right)$ and $d_2 = d_1 - \sigma\sqrt{T_0}$

With the definition that the survival probability at time T_0 is $\pi(T_0)$, it follows that

$$PL^*(0) = \pi(T_0) \cdot PL(0) \quad \text{and} \quad V^*(0) = \pi(T_0) \cdot V(0). \quad (2.14)$$

The option value thus is

$$V(0) = PL(0) \cdot [S(0)N(d_1) - KN(d_2)], \quad (2.15)$$

where

$$PL(0) = \sum_{i=1}^{n} D(0,T_i) \cdot (T_i - T_{i-1}) \int_0^{T_n} q(u)du + \int_0^{T_n} A(u) \cdot D(0,u) \cdot q(u)du$$

2.2.2. Brigo (2004)

Brigo (2004) introduces one filtration switching formula which relates the natural filtration G_t with the default-free filtration F_t.

$$E_Q\left[1_{\{\tau>T\}} \cdot Payoff|G_t\right] = \frac{1_{(\tau>t)}}{Q(\tau > T|F_t)} E_Q\left[1_{\{\tau>T\}} \cdot Payoff|F_t\right], \quad (2.16)$$

where $Q(\tau > T)$ is the survival probability at T under the risk-neutral measure Q.

With Equation (2.16), the formula for CDS spread shown in Equation (2.7) can be represented as

$$S_{0,n}(0)$$
$$= \frac{\sum_{i=1}^{n} E_Q\left[D(0,T_i)\cdot 1_{\{T_{i-1}<\tau\leq T_i\}}|F_0\right]\cdot(1-R)}{Q(\tau>0|F_0)\sum_{i=1}^{n}(T_i-T_{i-1})\cdot\bar{P}(0,T_i)}$$
$$= \frac{\sum_{i=1}^{n} E_Q\left[D(0,T_i)\cdot 1_{\{T_{i-1}<\tau\leq T_i\}}|F_0\right]\cdot(1-R)}{C_{0,n}(0)}, \qquad (2.17)$$

where $C_{0,n}(0) = Q(\tau>0|F_0)\sum_{i=1}^{n}(T_i-T_{i-1})\cdot\bar{P}(0,T_i)$.

Under certain measure $Q^{0,n}$ with the associated numeraire $C_{0,n}$, all asset prices form martingales. With the assumption of lognormal distribution, the dynamics of the CDS spread can be expressed as

$$\frac{dS_{0,n}(t)}{S_{0,n}(t)} = \sigma\cdot dW^{0,n}(t). \qquad (2.18)$$

Based on this framework, the value of the option under the measure $Q^{0,n}$ is

$$V(0) = C_{0,n}(0)\cdot E_{Q^{0,n}}\left[{V(T_0)}/{C_{0,n}(T_0)}\,|G_0\right]. \qquad (2.19)$$

With the filtration switching formula, the option value is

$$V(0) = \frac{C_{0,n}(0)}{Q(\tau>0|F_0)}\cdot E_{Q^{0,n}}\left[{V(T_0)}/{C_{0,n}(T_0)}\,|F_0\right]. \qquad (2.20)$$

Because $V(T_0) = C_{0,n}(T_0)\cdot max(S(T_0)-K,0)$ at T_0, Equation (2.20) implies

$$V(0)$$
$$= \frac{C_{0,n}(0)}{Q(\tau>0|F_0)}\cdot E_{Q^{0,n}}\left[max\left(S_{0,n}(T_0)-K,0\right)|F_0\right]$$
$$= \sum_{i=1}^{n}(T_i-T_{i-1})\cdot\bar{P}(0,T_i)\cdot E_{Q^{0,n}}\left[max\left(S_{0,n}(T_0)-K,0\right)|F_0\right]. \qquad (2.21)$$

According to Black formula (1976), the European CDS option thus can be represented as

$$V(0)=\sum_{i=1}^{n}(T_i-T_{i-1})\cdot\bar{P}(0,T_i)\cdot\left[S_{0,n}(0)N(d_1)-KN(d_2)\right],$$

where

$$d_1=\frac{1}{\sigma\sqrt{T_0}}\left(ln\left(\frac{S_{0,n}(0)}{K}\right)+\frac{1}{2}\sigma^2\cdot T_0\right)$$

and

$$d_2=d_1-\sigma\sqrt{T_0}.$$

2.3. Valuation method for American CDS options

Brigo (2006) proposes a dynamic programming to price an American CDS option. The default of the reference company is based on a doubly stochastic framework and the default intensities are assumed to follow CIR++ process. The parameters of the model are estimated by a cross sectional calibration-based method and a historical estimation approach.

However, this method is based on a very complex mathematical algebra thus is difficult and not intuitive to understand. In the following paper, we use the one-period CDS spread model presented by Brigo (2004) to value an American CDS option. The main advantage is that the one-period CDS spread model is similar with LIBOR market model in interest theories. The framework is to simulate the one-period forward CDS spreads founded on specific spread dynamics and calculate the option prices with least-squares method. Both concepts are on the basis of well-known ideas so that the valuation can be easily completed.