III. Valuation Framework for CDS options

In simulation, the underlying asset price is the most important variable. The suitable dynamics is selected to describe the underlying spreads. The relevant parameters such as volatilities and correlations in the dynamics are calibrated with market data. The simulated paths of the asset prices are obtained by repeatedly drawing random numbers. Finally, the option values can be easily calculated by averaging the simulated values of the paths.

As for a CDS option, its underlying asset is a CDS contract and the primary variable to determine the value of a CDS contract is CDS spread. To get the value of a CDS option, we need to simulate the CDS spread. However, instead of directly simulating CDS spreads, we simulate forward one-period CDS spreads which can be stripped out from regular quotes for CDS contracts.

The dynamics to be used in this paper are from the one-period forward CDS spread model presented by Brigo (2005). The main idea of this model is to convert the market quotes for CDS contracts to forward one-period CDS spreads, and simulate these spreads with specific dynamics. To get simulated multi-period CDS spread, we need only to substitute the simulated one-period CDS spreads with certain formula which will be introduced later.

To get the simulated value of a European CDS option, we only need to find out the option prices at the maturity with the simulated CDS spreads, and discount these prices to obtain the option value. For an American option, the procedure may be a little more complicated. We have to compute the values of the Americans option along each path by using least-squares method.

The primary advantage of this model is that it is similar with LIBOR market model in interest rate theories. The one-period CDS spread model uses forward spreads as main variables , and assumes the forward one-period CDS spread is a martingale and log-normal distributed under respective probability measure. These concepts are the same as those in LIBOR market model. As long as the basic ideas about LIBOR market model are realized, path-dependent CDS-related products can be easily priced.

In this chapter, we first introduce forward one-period CDS spread and the one-period CDS spread model. Secondly we present how we apply this model to a European CDS option. Finally, we detail how we price an American CDS option with this model.

3.1. Dynamics of one-period forward CDS spreads

The definition of one-period forward CDS spreads is similar with that of forward interest rates. A one-period forward CDS spread which is alive for three month in one month from now can be denoted by *S(0;1m,3m)*. The first number zero in the bracket represents the current time, and the *1m* and *3m* mean the starting and ending time of a one-period forward CDS spread repeatedly. The starting and ending time of the spread do not change with the passage of time. Once the *1m* is reached, the forward CDS spread one month ago now becomes a spot CDS spread.

The only difference between forward interest rates and one-period forward CDS spreads is that market quotes for one-period forward CDS spreads are still not universal. The prevalent quotes for CDS contracts are usually spot spreads. To obtain one-period forward CDS spreads, we have to strip these market CDS quotes with certain formula.

An example is given to explain how to reach one-period CDS spreads. The one-period forward CDS spreads from now (time 0) to time T for a reference company are denoted by $\hat{S}_1(0), \hat{S}_2(0) \dots, \hat{S}_M(0)$, where $T_1 < T_2 < \cdots < T_M$.[2] The k^{th} spread can be computed by[3]

$$\hat{S}_k(0) = (1 - R)\frac{\bar{P}(0, T_{k-1}) \cdot P(0, T_k)/P(0, T_{k-1}) - \bar{P}(0, T_k)}{(T_k - T_{k-1}) \cdot \bar{P}(0, T_k)}. \tag{3.1}$$

As Equation (3.1) shows, the one-period CDS spread is mainly composed of risk-free and corporate zero coupon bonds. The risk-free zero coupon bonds can be obtained by zero curve which is usually stripped from interest rate swaps. As for the corporate zero coupon bonds in Equation (3.1), we do not replace them with bond prices quoted in the market. This is primarily because not all corporate bonds for the reference company are liquid enough to reflect the true value of bonds. The bias may arise when we estimate the zero curve for the reference company. Therefore, inserting such data to our model may produce unrealistic results.

To deal with this problem, we first recall from the following equation

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[2] For clarity, we simplify the notation $S(0; T_0, T_1), S(0; T_1, T_2) \dots, S(0; T_{M-1}, T_M)$ as $\hat{S}_1(0), \hat{S}_2(0) \dots, \hat{S}_M(0)$.

[3] For details, please see Brigo (2006).

$$\bar{P}(0,T_k) = E_Q\left[D(0,T_k)\cdot 1_{\{\tau>T_k\}}|G_0\right].$$

Under the assumption of independence of interest rates and default, this can be

$$\bar{P}(0,T_k) = E_Q[D(0,T_k)|G_0]\cdot E_Q\left[1_{\{\tau>T_k\}}|G_0\right]$$

$$= P(0,T_k)\cdot Q(\tau > T_k|G_0). \qquad (3.2)$$

The survival probabilities can be calculated from market quotes for CDS contracts on the reference company.[4] Given the zero curve for interest rates and survival probabilities for the reference company, the values of corporate zero coupon bonds can be calculated. Therefore, we can obtain the set of one-period forward CDS spreads with Equation (3.1).

Since one-period forward CDS spreads have been reached, we now discuss how the dynamics are derived. Changed to one-period CDS spread expression, Equation (2.15) can be rewritten as follows

$$\hat{S}_k(0) = \frac{E_Q\left[D(0,T_k)\cdot 1_{\{T_{k-1}<\tau\leq T_k\}}|F_0\right]\cdot(1-R)}{\hat{C}_k(0)}, \qquad (3.3)$$

where $\hat{C}_k(0)$ means $C_{k-1,k}(0)$.

Following the concept in the section 2.2.2, we set $\hat{C}_k$ as the numeraire. The dynamics for the one-period CDS spread under the probability measure $\hat{Q}^k$ can be

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[4] In this paper, we adopt the reduced form model to match the survival probability, and the default intensity function is assumed to be piece-wise constant.

assumed to be

$$\frac{d\hat{S}_k(t)}{\hat{S}_k(t)} = \sigma_k \cdot d\widehat{W}_k^k(t) \qquad (3.4)$$

where σ_k is the instantaneous volatility for the k^{th} one-period CDS spread, $\widehat{W}^k$ is the Brownian motion under the measure $\hat{Q}^k$.

Equation (3.4) means that the dynamics for each one-period CDS forward spread are log-normal distributed under their respective probability measures. However, these dynamics are meaningful to valuation only when all the probability measures are changed to identical measure. Thus Equation (3.4) has to be further derived.

Suppose now there is a CDS-related product involving two one-period CDS spreads, $\hat{S}_1(t)$ and $\hat{S}_2(t)$. Both dynamics for $\hat{S}_1(t)$ and $\hat{S}_2(t)$ are martingales and log-normal distributed under the respective measures $\hat{Q}^1$ and $\hat{Q}^2$. For meaningful valuation, the probability measures have to be consistent.

With the formula for change of numeraire by Brigo (2006), we can overcome the problem of different probability measures. The Brownian motion for $\hat{S}_1(t)$ under the measure $\hat{Q}^2$can be expressed as

$$d\widehat{W}_1^1(t) = d\widehat{W}_1^2(t) + \rho_{1,2} \cdot dln\left(\hat{S}_1(t)\right) \cdot dln\left(\frac{\hat{C}_1(t)}{\hat{C}_2(t)}\right) \qquad (3.5)$$

where $\rho_{1,2}$ is the instantaneous correlation between $\hat{S}_1(t)$ and $\hat{S}_2(t)$.

Therefore, the dynamics for $\hat{S}_1(t)$ under the measure $\hat{Q}^2$thus is

$$\frac{d\hat{S}_1(t)}{\hat{S}_1(t)} = -\sigma_1 \frac{\rho_{1,2} \cdot \sigma_2 \cdot S_2(t) \cdot (T_2 - T_1)}{\hat{S}_2(t) \cdot (T_2 - T_1) + (1 - R)} dt + \sigma_2 \cdot d\widehat{W}_1^2(t) \qquad (3.6)$$

In practice, there are usually more than two dynamics when a CDS-related product is valued. This means Equation (3.6) has to be further generalized for valuation. Suppose we split a particular time spanning over the maturity of the CDS-related product into m periods, $t_0, t_1, t_2, \ldots, t_m$. With the same procedure described above, the k^{th} dynamics for one-period spread under the measure $\hat{Q}^m$ is

$$\frac{d\hat{S}_i(t)}{\hat{S}_i(t)} = -\sigma_i \sum_{k=i+1}^{m} \frac{\rho_{i,k} \cdot \sigma_k \cdot \hat{S}_k(t)(T_k - T_{k-1})}{\hat{S}_k(t) \cdot (T_k - T_{k-1}) + (1-R)} dt + \sigma_i \cdot d\widehat{W}_i^m(t) \qquad (3.7)$$

Now that we have reached a consistent probability measure for the dynamics of the one-period CDS spreads, the next step is to derive a feasible formula for simulation. By Ito-Doeblin formula[5], Equation (3.4) can be arranged as

$$dln\left(\hat{S}_i(t)\right) = \left[-\sigma_i \sum_{k=i+1}^{m} \frac{\rho_{i,k} \cdot \sigma_k \cdot \hat{S}_k(t)(T_k - T_{k-1})}{\hat{S}_k(t) \cdot (T_k - T_{k-1}) + (1-R)} - \frac{1}{2}\sigma_i^2\right] dt + \sigma_i \cdot d\widehat{W}_i^m(t) \qquad (3.8)$$

With a little algebra, Equation (3.8) becomes

$$\begin{aligned} &ln\left(\hat{S}_i(t+\Delta t)\right) \\ &= ln\left(\hat{S}_i(0t)\right) + \left[-\sigma_i \sum_{k=i+1}^{m} \frac{\rho_{i,k} \cdot \sigma_k \cdot \hat{S}_k(t) \cdot (T_k - T_{k-1})}{\hat{S}_k(t) \cdot (T_k - T_{k-1}) + (1-R)} - \frac{1}{2}\sigma_i^2\right] \Delta t \\ &\qquad + \sigma_i \left[\widehat{W}_i^n(t+\Delta t) - \widehat{W}_i^n(t)\right] \end{aligned} \qquad (3.9)$$

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[5] For more details, please see Steven E. Shreve (2000).

Equation (3.9) is the formula dealing with Monte Carlo simulation for one-period CDS spreads. This formula is almost the same with the formula for LIBOR market model in interest rate theory except that the denominator in Equation (3.9) appears a recovery rate term. Another difference behind the formula is that the market quotes for forward CDS spreads are not prevalent. To apply this model to the valuation, we have to strip out one-period forward spreads from market quotes for CDS contracts. After inserting these estimated spreads, we can start a Monte Carlo simulation.

3.2. Valuation framework for European CDS options

Before simulating American CDS options, we have to notice that all simulated spreads resulting from Equation (3.9) are of one-period length. For holders of American CDS options, they exercise options on the basis of whether the CDS "spot spread" is greater than the exercise price. In other words, the holders' decisions primarily depend on multi-period spreads instead of one-period spreads. Therefore, this means that the simulated one-period spreads have to be converted to multi-period spreads so that the valuation is consistent with the market convention.

Brigo (2006) derives a formula relating these two types of CDS spreads as

$$S_{0,m}(t) \approx \sum_{i=1}^{m} w_i \cdot \hat{S}_i(t) \tag{3.10}$$

where

$$w_i = (t_i - t_{i-1}) \cdot \bar{P}(0,t_i) \Big/ \sum_{i=1}^{m} (t_i - t_{i-1}) \cdot \bar{P}(0,t_i).$$

The term w_i in Equation (3.10) can be seen as the weight of i^{th} one-period spread and is composed of a set of corporate zero coupon bonds observed at the initial time.

Now that Equation (3.10) has provided us an idea of how to convert one-period simulated CDS spreads to multi-period simulated CDS spreads, we start to calculate the value of a European CDS option under this simulation framework. Suppose a European CDS option matures at time T_0, and the underlying forward CDS contract starts from T_0 to T_n, in which the protection premiums are paid at T_1, T_2,…T_{n-1}, T_n. Recalled from Equation (2.19), the option value under the measure $Q^{0,n}$ is expressed as

$$V(0) = \sum_{i=1}^{n} (T_i - T_{i-1}) \cdot \bar{P}(0,T_i) \cdot E_{Q^{0,n}}\left[max\big(S_{0,n}(T_0) - K, 0\big)|F_0\right] \qquad (3.11)$$

For simulation purpose, we split the contract life from T_0 to T_n into m periods, t_0= T_0, t_1, t_2,…, t_m= T_n, and set $\hat{C}_m$ as the valuation numeraire. Thus the option value under the measure $\hat{Q}^m$ is[6]

$$V(0) = \bar{P}(0,T_m) \cdot E_{\hat{Q}^m}\left[\frac{\sum_{i=1}^{n}(T_i - T_{i-1}) \cdot \bar{P}(T_0,T_i)}{\bar{P}(T_0,T_m)} max\big(S_{0,n}(T_0) - K, 0\big)|F_0\right]$$

(3.12)

6 For more details, please see the appendix.

If we shift from the initial time to the option maturity T_0, the option value appears to be

$$V(T_0) = \sum_{i=1}^{n} (T_i - T_{i-1}) \cdot \bar{P}(T_0, T_i) \cdot max\left(S_{0,n}(T_0) - K, 0\right) \tag{3.13}$$

Apparently, Equation (3.13) represents the option value at T_0. The summation term explains that there are *n* cash flows of protection payments on the CDS contract once the option is exercised.

In credit models, one of the most difficult parts is to describe a default event in mathematics. Asset values always dramatically change with the occurrence of a default. In spite of this, Equation (3.12) provides an easy way to value a European CDS option. We simulate CDS spreads at the option maturity, insert these spreads in Equation (3.13) to obtain the immediate exercise value, and reach the initial option values with Equation (3.12). It is unnecessary to consider the default event in simulation because this is implicitly included in the probability measure $\hat{Q}^m$. Intuitively speaking, the default information is actually contained in the corporate zero coupon bonds in Equation (3.12). Consequently, we only need to focus on the variations of the CDS spreads in simulating European CDS options.

3.3. Valuation framework for American CDS options with least-squares Monte Carlo simulation

Although the basic concepts of valuing American CDS options are much the same with those of valuing European CDS options, there are still some differences between them. The clearest distinction is that the life of underlying CDS contract for an American CDS option varies as the holder exercises at different moments. For example, suppose the option maturity and the protection maturity for an American CDS option are 1 and 2 years respectively. When the option is exercised at the end of the six months, the holder will receive a CDS contract of which the life is four and a half years. This means the holder has to make periodical payments for four and a half years. Analogously, the life of the CDS contract is 4 years when the option is exercised at the end of one year. Consequently, the underlying CDS contract depends mainly on the time at which the holder exercises.

To deal with this situation, let us further take an example to show how we apply the one-period spread model. Suppose a Bermudan CDS option with option maturity T_0 and protection maturity T_n is exercisable at $T_1^{'}, T_2^{'} \ldots T_n^{'} = T_0$. As mentioned above, the underlying CDS life is from $T_1^{'}$ to T_n when this option is exercised at $T_1^{'}$.

Because Bermudan CDS option has the early-exercise characteristic, we have to start from the moment that the Bermudan CDS option matures and recursively calculate the simulated value. Suppose we simulate h paths of one-period CDS spreads for the underlying CDS contract. With Equation (3.10), the simulated multi-period CDS spreads at the option maturity, $S_1^{T_n^{'},T_n}(T_n^{'}), S_2^{T_n^{'},T_n}(T_n^{'}) \ldots S_h^{T_n^{'},T_n}(T_n^{'})$, can be calculated from these simulated one-period CDS spreads. Using these simulated multi-period CDS spreads, we can get the corresponding immediate exercise values, $EV_1(T_n^{'})$, $EV_2(T_n^{'}) \ldots$, $EV_h(T_n^{'})$, with Equation (3.13).

Only these immediate exercise values are not enough for us to find the value of the Bermudan option. We have to further look for the relationship of option values among exercisable moments so that the option value can be recursively calculated. The following formula relates the option value at $T_k^{'}$ with that at $T_{k-1}^{'}$[7].

$$V(T_{k-1}^{'}) = \bar{P}(T_{k-1}^{'}, T_m) \cdot E_{Q^m}\left[\frac{V(T_k^{'})}{\bar{P}(T_k^{'}, T_m)} | F_{T_k^{'}}\right] \tag{3.14}$$

With Equation (3.14), the option values at $T_{n-1}^{'}$, $V_1(T_{n-1}^{'}), V_2(T_{n-1}^{'})\ldots, V_h(T_{n-1}^{'})$, can be calculated from the corresponding immediate exercise values, $EV_1(T_n^{'}),\ EV_2(T_n^{'})\ldots, EV_h(T_n^{'})$.

Let's further combine the above procedure with least-squares regression. We refer to the simulated multi-period spreads at $T_{n-1}^{'}$ as the independent variable, and the option values at $T_{n-1}^{'}$ are set as dependent variable. Then we regress the dependent variable on the independent variable according to the following regression model.

$$V_i(T_k^{'}) = \beta_0 \cdot L_0\left(S_i^{T_k^{'},T_n}(T_k^{'})\right) + \beta_1 \cdot L_1\left(S_i^{T_k^{'},T_n}(T_k^{'})\right) + \beta_2 \cdot L_2\left(S_i^{T_k^{'},T_n}(T_k^{'})\right)$$

$$for\ i = 1,2\ \ldots;\ \ k=1,2\ldots n\text{-}1 \tag{3.15}$$

where

$$L_0\left(S_i^{T_k^{'},T_n}(T_k^{'})\right) = exp\left(-S_i^{T_k^{'},T_n}(T_k^{'})/2\right)$$

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[7] This formula provides an idea of discount similar with the money market account in risk-neutral measure. The difference is that we use corporate zero coupon bonds as our discount factor here.

$$L_1\left(S_i^{T_k',T_n}(T_k')\right)=exp\left(-S_i^{T_k',T_n}(T_k')/2\right)\cdot\left(1-S_i^{T_k',T_n}(T_k')\right)$$

$$L_2\left(S_i^{T_k',T_n}(T_k')\right)=exp\left(-S_i^{T_k',T_n}(T_k')/2\right)\cdot\left(1-2S_i^{T_k',T_n}(T_k')+\left(S_i^{T_k',T_n}(T_k')\right)^2\right)$$

Thus a set of estimated option values at T_{n-1}', $\hat{V}_1(T_{n-1}')$, $\hat{V}_2(T_{n-1}')\ldots$, $\hat{V}_h(T_{n-1}')$, are obtained. The immediate exercise value on each path, $EV_1(T_{n-1}')$, $EV_2(T_{n-1}')\ldots$, $EV_h(T_{n-1}')$, is then compared with the corresponding estimated option value. Once the exercise value is greater, the option is exercised at T_{n-1}'. Consequently, repeating these steps recursively until the initial time, we can determine the optimal exercise time on each path. With Equation (3.14), the Bermudan option thus can be valued by discounting these cash flows of which the option is optimally exercised.[8]

For an American CDS option, we only need increase the number of the exercisable moments. The more the exercisable moments are, the more accurate the value of an American CDS option is.

Now that a complete simulation of American CDS options can be implemented, let us specify our procedure to price American CDS options.

1. Strip out the implied intensity from market quotes for CDS contracts, and calculate survival probabilities during protection maturity of an option so that corporate zero coupon bonds can be reached.

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[8] This method is known as least-squares approach. For more details, please see Longstaff and Schwartz (2001).

2. Strip out one-period CDS spreads by Equation (3.1), and calculate historical volatilities and correlations with these one-period CDS spreads to calibrate for simulation.[9]
3. Simulate one-period CDS spreads with Equation (3.9), and convert them to desired multi-period CDS spreads.
4. Calculate immediate exercise values during the option maturity with Equation (3.13).
5. Calculate recursively initial values of American CDS options with least-squares approach.

[9] We use historical volatilities and correlations as our proxy because the market for European CDS options is not prevalent. Brigo (2006) provides a formula dealing with calibration for correlation of one-period CDS spreads under the situation that the market is prevalent.