A NOTE ON THE REVISED SIMPLEX METHOD FOR SOLVING A MATRIX GAME

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1. INTRODUCTION

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In the game of matching pennies, player row (R) chooses "heads" (H) or "tails" (T). Player column (C), not knowing player R's choice, also chooses (H) or (T). If what they choose is alike, player C wins one dollar from player R; otherwise, player R wins one dollar from player C. Obviously, each player has two strategies "H" and "T", and the payoff of this game is in form as follows:

Player C

		H	T
Player R	H	(-1, 1)	(1, -1)
	T	(1, -1)	(-1, 1)

(1,1)

where the sum of each ordered pair is zero, called zero-sum game. Hence, there are one payoff matrix A to player R and another payoff matrix B to player C,

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (1, 2)$$

For the zero-sum case, $A = -B$ (1, p. 352), anything won by one player is lost by another and the payoff matrix A, called a matrix game, can be used to explain player C's winnings as negative and losses as positive payoff. If the sum of at least one ordered pair (1,1) is not zero, called the nonzerosum case, the two payoff matrices A and B are independent.

Consider a game (two-person, player R and player C, and zero-sum with a finite number of strategies) with an m by n payoff matrix $A = (a_{ij})_{(m,\ n)}$. Suppose player R chooses the ith strategy or ith row of A and player C the jth strategy or jth column of A, player C is to pay player R an amount a_{ij}. Since this game is zero-sum, the payoff to player C is $(-a_{ij})$. Player R is trying to make a_{ij}

as large as possible, hence, called the maximum player, whereas player C on the other hand wants to be $(-a_{ij})$ as large as possible or a_{ij} as small as possible, henc, called the minimum player.

To solve a matrix game, it is provided that player will want to maximize his expected minimum winnings and player C to minimize his expected maximum losses. Thus we have the minimax or maximin theorem (2, P. 635). Hence, it is defined in the form as follows:

Player C

$$
\text{Player R}\quad
\begin{array}{c}
 \\ x_1 \\ x_2 \\ x_3 \\ \\ x_i \\ \\ x_m \\ \\ \\
\end{array}
\begin{array}{ccccccc}
y_1 & y_2 & \cdots & y_j & \cdots & y_n & \\
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & \min_j a_{1j} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & \min_j a_{2j} \\
a_{31} & a_{32} & \cdots & a_{3j} & \cdots & a_{3n} & \min_j a_{3j} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & \min_j a_{ij} \quad \max_i \min_j a_{ij} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & \min_j a_{mj} \\
\max_i a_{i1} & \max_i a_{i2} & & \max_i a_{ij} & & \max_i a_{in} & \\
 & & & \min_j \max_i a_{ij} & & & \\
\end{array}
\qquad (1,3)
$$

Now for any strategy i which player R may choose, no matter what strategy player C chooses, he can be sure to get the expected minimum payoff, $\min_j a_{ij}$. As player is at liberty to choose i, he will therefore want to maximize his expected minimum winnings that is to get at least.

$$\max_i \min_j a_{ij}. \qquad (1,4)$$

Similarly, for any strategy j which player C may choose, no matter what strategy Player R chooses, he can be sure to get the expected minimum payoff, $\min_i (-a_{ij})$, or; equivalently, to lose his expected maximum payoff, $\max_i a_{ij}$. $[\min_i (-a_{ij}) = -\max a_{ij}]$ As player C is at liberty to choose j, he will therefore want to minimize his expected maximum losses or to pay player R at most,

$$\min_j \max_i a_{ij}. \qquad (1,5)$$

In general, the two quantities (1,4) and (1,5) might be different but would satisfy the following relationship (3, P. 117).

$$\max_i \ \min_j \ a_{ij} \leqslant \ \min_j \max_i a_{ij} \qquad (1,6)$$

Example 1. Suppose the game is such that its payoff matrix $A = (a_{ij})_{(3,4)}$ is given by

		Player C			
Player R	2	3	4	-1	$\min_4 a_{14} = -1$
	-2	5	-3	-1	$\min_3 a_{23} = -3$
	4	1	3	2	$\min_2 a_{32} = 1$
	$\max_3 a_{31} = 4$	$\max_2 a_{22} = 5$	$\max_1 a_{13} = 4$	$\max_3 a_{34} = 2$	

then $\max_3 \min_2 a_{32} = 1$ and $\min_4 \max_3 a_{34} = 2$. In this game, player R can get at least 1 from player C and player C may pay player R at most 2; therefore player R will try to get more that 1 and player C to lose 2, or, equivalently, player R would win his expected payoff between 1 and 2.

If the relationship (1, 6) happens to be an equality:

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = V, \qquad (1,7)$$

player R could choose a strategy so as to get at least the common value V and player C employs a strategy to lose at most V. For this case, there are two strategies i′ and j′ for player R and player C such that for all j and i respectively

$$a_{ij'} \leqslant a_{i'j'} \leqslant a_{ij}, \qquad (1,8)$$

where $a_{i'j'} = V$ is called the value of the matrix game $A = [a_{ij}]_{(m,n)}$ to player R from player C. The strategies i′ and j′ are called their optimal strategies (soution to the matrix game) and also named a saddle point (i′ j′).

Exaple 2. The game, given by the payoff matrix $A = [a_{ij}]_{(3, 4)}$

		Player C			
Player R	2	-2	4	-1	$\min_2 a_{12} = -2$
	-2	5	-3	-1	$\min_3 a_{23} = -3$
	4	6	3	2	$\min_4 a_{34} = 2$
	$\max_3 a_{31} = 4$	$\max_3 a_{32} = 6$	$\max_1 a_{13} = 4$	$\max_3 a_{34} = 2$	

has a saddle point (i', j') = (3, 4) because $\max_3 \min_4 a_{34} = \min_4 \max_3 a_{34} = 2$. It is clear that player R can get his expected payoff from player C at least 2 whereas player C can keep what player R would get from exceeding 2. The strategies i' = 3 and j' = 4 are the optimal solution for the matrix game $A = [a_{ij}](3,4)$ and its value is 2.

If a matrix game has a saddle point, it will be solved as in example 2; otherwise, as in exmaple 1 it is not.

2. TO REDUCE A MATRIX GAME TO A PROBLEM OF LINEAR PROGRAMMING

To solve a matrix game $A = [a_{ij}]_{(m,\ n)}$ which has no saddle point is equivalent to solving a porblem of linear programming and therefore, a matrix game can be converted into a problem of linear programming (4, P. 252, and 5, P. 409). Let the m by 1 probability column vector $X^T = (x_1\ x_2\ x_3\ \dots\ x_i\ \dots\ x_m)$ be player R's mixed strategy, where $\sum_{i=1}^{m} x_i = 1$, $x_i \geqslant 0$ for all i and the n by 1 probability column vector $Y^T = (y_1\ y_2\ y_3\ \dots\ y_j\ \dots\ y_n)$ be player C's mixed strategy, where $\sum_{j=1}^{n} y_j = 1$, $y_j \geqslant 0$ for all j. If player C chooses strategies y_j (j = 1, 2, 3, ..., n), and player R takes the mixed strategy $X^T = (x_1\ x_2\ x_3\ \dots\ x_9\ \dots\ x_m)$, then player R can get the expected payoff, $\sum_{i=1}^{m} a_{ij}x_i$, for all j, that is at least the game value V. Hence, the following equations are obtained

$$
\begin{array}{lllll}
x_1 & + \quad x_2 & + \quad x_3 & + \dots + & x_m = 1 \\
a_{11}x_1 & + a_{21}x_2 & + a_{31}x_3 & + \dots + & a_{m1}x_m \geqslant V \\
a_{12}x_1 & + a_{22}x_2 & + a_{32}x_3 & + \dots + & a_{m2}x_m \geqslant V \\
a_{13}x_1 & + a_{23}x_2 & + a_{33}x_3 & + \dots + & a_{m3}x_m \geqslant V \\
\dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots \\
a_{1n}x_1 & + a_{2n}x_2 & + a_{3n}x_3 & + \dots + & a_{mn}x_m \geqslant V
\end{array}
\qquad (2,1)
$$

which is equivalent to the matrix form

$$
\begin{array}{l}
\beta^T X = 1 \\
A^T X \geqq \alpha V,
\end{array}
\qquad (2,2)
$$

where α and β are respectively the n by 1 and the m by 1 matrices with ones.

Similarly, if player R chooses strategies x_i (i = 1, 2, 3, ..., m), and player C takes the mixed strategy $Y^T = (y_1\ y_2\ y_3\ \dots\ y_j\ \dots\ y_n)$, then he will pay player R's expected payoff $\sum_{j=1}^{n} a_{ij}y_j$, for all i, that is at most the game value V. Hence, the following equations are obtained

$$\begin{aligned}
y_1 + y_2 + y_3 + \ldots + y_n &= 1 \\
a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \ldots + a_{1n}y_n &\leqslant V \\
a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \ldots + a_{2n}y_n &\leqslant V \\
a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + \ldots + a_{3n}y_n &\leqslant V \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
a_{m1}y_1 + a_{m2}y_2 + a_{m3}y_3 + \ldots + a_{mn}y_n &\leqslant V
\end{aligned} \qquad (2,3)$$

which is equivalent to the matrix form

$$\begin{aligned}
\alpha^T Y &= 1 \\
AY &\leqslant \beta V.
\end{aligned} \qquad (2,4)$$

From (2,1), player R will want to maximize V or equivalently, to minimize $\frac{1}{V}$ and from (2, 3), player C to minimize V or to maximize $\frac{1}{V}$. Let $p_i = \frac{1}{V} x_i$ for all i be the components of the column vector $P^T = (p_1\ p_2\ p_3\ \ldots\ p_i\ \ldots\ p_m)$ and $q_j = \frac{1}{V} y_{ij}$ for all j the components of the column fector $Q^T = (q_1\ q_2\ q_3\ \ldots\ q_j\ \ldots\ q_n)$, then the equations (2, 1) and (2, 3) can respectively be converted into the following problems of linear programming (5, P. 409).

Minimize $\quad \frac{1}{V} = p_1 + p_2 + p_3 + \ldots + p_m$

Subject to

$$\begin{aligned}
a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + \ldots + a_{m1}p_m &\geqslant 1 \\
a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + \ldots + a_{m2}p_m &\geqslant 1 \\
a_{13}p_1 + a_{23}p_2 + a_{33}p_3 + \ldots + a_{m3}p_m &\geqslant 1 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
a_{1n}p_1 + a_{2n}p2 + a_{3n}p_3 + \ldots + a_{mn}p_m &\geqslant 1 \\
p_i \geqslant 0, \text{ for all } i, &
\end{aligned} \qquad (2,5)$$

and Maximize $\quad \frac{1}{V} = q_1 + q_2 + q_3 + \ldots + q_n$

Subject to

$$\begin{aligned}
a_{11}q_1 + a_{12}q_2 + a_{13}q_3 + \ldots + a_{1n}q_n &\leqslant 1 \\
a_{21}q_1 + a_{22}q_2 + a_{23}q_3 + \ldots + a_{2n}q_n &\leqslant 1 \\
a_{31}q_1 + a_{32}q_2 + a_{33}q_3 + \ldots + a_{3n}q_n &\leqslant 1 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
a_{m1}q_1 + a_{m2}q_2 + a_{m3}q_3 + \ldots + a_{mn}q_n &\leqslant 1 \\
q_j \geqslant 0, \text{ for all } j. &
\end{aligned} \qquad (2,6)$$

By using simplex method (algorithm), the optimal solutions to these two problems, (2, 5) and (2, 6), can be solved and they are precisely in the dual forms according to the duality theorem (6, P. 219). If its value V is zero or negative, these two problems can not be defined and obviously, the value, V, must be positive. Hence, the objective of this note is to modify the problem of linear programming and revise the simplex method so as to find the optimal solution for the matrix game no matter whether the game value, V, is positive, zero, or negative.

3. RESULTS

Because player R wants to maximize V and player C to minimize V, the problems (2, 1) and (2, 3) can be defined respectively as follows:

$$
\begin{array}{ll}
\text{Maximize} & f(x_1, x_2, x_3, \ldots, x_m, V) = V \\
\text{Subject to} & -a_{11}x_1 - a_{21}x_2 - a_{31}x_3 - \ldots - a_{m1}x_m + V \leqslant 0 \\
& -a_{12}x_1 - a_{22}x_2 - a_{32}x_3 - \ldots - a_{m2}x_m + V \leqslant 0 \\
& -a_{13}x_1 - a_{23}x_2 - a_{33}x_3 - \ldots - a_{m3}x_m + V \leqslant 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& -a_{1n}x_1 - a_{2n}x_2 - a_{3n}x_3 - \ldots - a_{mn}x_m + V \leqslant 0 \\
& x_1 + x_2 + x_3 + \ldots + x_m = 1 \\
& x_i \geqslant 0, \text{ for all i and V unrestricted in sign,}
\end{array}
\qquad (3,1)
$$

or

$$
\begin{array}{ll}
\text{Maximize} & V \\
\text{Xubject to} & -A^T X + \alpha V \leqslant 0 \\
& \beta^T X = 1 \\
& X \geqslant 0 \text{ and V unrestricted in sign,}
\end{array}
\qquad (3,2)
$$

and

$$
\begin{array}{ll}
\text{Minimize} & g(y_1, y_2, y_3, \ldots, y_n, V) = V \\
\text{Subject to} & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \ldots + a_{1n}y_n - V \leqslant 0 \\
& a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \ldots + a_{2n}y_n - V \leqslant 0 \\
& a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + \ldots + a_{3n}y_n - V \leqslant 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& a_{m1}y_1 + a_{m2}y_2 + a_{m3}y_3 + \ldots + a_{mn}y_n - V \leqslant 0 \\
& y_1 + y_2 + y_3 + \ldots + y_n = 1 \\
& y_j \geqslant 0, \text{ for all j and V unrestricted in sign,}
\end{array}
\qquad (3,3)
$$

or Minimize V

Subject to $AY-\beta V \leqslant 0$

$\alpha^T Y = 1$

$Y \geqslant 0$ and V unrestricted in sign. (3,4)

The problems (3, 2) and (3, 4) are degenerated from the bimatrix game (nonzero-sum case).

By using simplex method, the n+1 non-negative variables, $u_k \geqslant 0$ (k=1, 2, 3, ..., n, n+1), usually called slack variables, must be introduced to change (3, 1) into the following form

maximize $f(x_1, x_2, x_3, ..., x_i, ..., x_m, V) = V$

Subject to

$$
\begin{array}{l}
-a_{11}x_i-a_{21}x_2 \cdots -a_{i1}x_i \cdots -a_{m1}x_m + V + u_1 = 0 \\
-a_{12}x_i-a_{22}x_2 \cdots -a_{i2}x_i \cdots -a_{m2}x_m + V + u_2 = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-a_{1j}x_1-a_{2j}x_2 \cdots -a_{ij}x_i \cdots -a_{mj}x_m + V + u_j = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-a_{1n}x_1-a_{2n}x_2 \cdots -a_{in}x_i \cdots -a_{mn}x_m + V + u_n = 0 \\
\quad x_1 + \quad x_2 \cdots + \ x_i \cdots + \quad x_m + u_{n+1} = 1 \\
\quad x_i \geqslant 0 \text{ for all i, } u_k \geqslant - \text{ for all k and V unrestricted in sign.}
\end{array}
\qquad (3,5)
$$

From (3, 5), the simplex matrix of the matrix game $A = [a_{ij}]_{(m,\ n)}$ is defined as follows:

$$
\left(
\begin{array}{cccccccc|cccccccc|c}
x_1 & x_2 & \cdots & x_i & \cdots & x_m & V & & u_1 & u_2 & \cdots & u_j & \cdots & u_n & u_{n+1} & & \\
-a_{11} & -a_{21} & \cdots & -a_{i1} & \cdots & -a_{m1} & 1 & & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & & 0 \\
-a_{12} & -a_{22} & \cdots & -a_{i2} & \cdots & -a_{m2} & 1 & & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
-a_{1j} & -a_{2j} & \cdots & -a_{ij} & \cdots & -a_{mj} & 1 & & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
-a_{1n} & -a_{2n} & \cdots & -a_{in} & \cdots & -a_{mn} & 1 & & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & & 0 \\
1 & 1 & \cdots & 1 & \cdots & 1 & 0 & & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & & 1 \\
\hline
0 & 0 & \cdots & 0 & \cdots & 0 & 1 & & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & & f
\end{array}
\right)
\qquad (3,6)
$$

Before applying the simplex method (6, P. 205) to find the optimal solution of the maximum problem (3, 5), the simplex matrix (3, 6) must be adjusted, and thus, we must make clear three things as follows:

1) The non-basic variable V must first be made basic instead of being one among the basic variables u_k, k = 1, 2, 3, ..., n.

2) The basic variable u_{n+1} must first be made non-basic instead of being one among the non-basic variables x_i, i = 1, 2, 3, ..., m.

3) After (1) and (2), all elements in the last column of the adjusted matrix (3, 7) must be non-negative except the jth element and the last one.

From (1, 4), let a_{ij} be the player R's expected maximizing minimum payoff, then he wants to maximize V starting from a_{ij}. Hence, (1), V becomes basic instead of u_j and (2), u_{n+1} becomes non-basic instead of x_i. The matrix (3, 6) can be adjusted by using elementary matrix row operations and we obtain

$$
\begin{array}{ccccccc|cccccccc|c}
x_1 & x_2 & \cdots & x_i & \cdots & x_m & V & u_1 & u_2 & \cdots & u_j & \cdots & u_n & u_{n+1} & & \\
b_{11} & b_{21} & \cdots & 0 & \cdots & b_{m1} & 0 & 1 & 0 & \cdots & -1 & \cdots & 0 & a_{ij}-a_{ij} & & a_{ij}-a_{ij} \\
b_{12} & b_{22} & \cdots & 0 & \cdots & b_{m2} & 0 & 0 & 1 & \cdots & -1 & \cdots & 0 & a_{i2}-a_{ij} & & a_{i2}-a_{ij} \\
\cdots & & & & & & & \cdots & & & & & & & & \\
d_1 & d_2 & \cdots & 0 & \cdots & d_m & 1 & 0 & 0 & \cdots & 1 & \cdots & & a_{ij} & & a_{ij} \\
\cdots & & & & & & & \cdots & & & & & & & & \\
b_{1n} & b_{2n} & \cdots & 0 & \cdots & b_{mn} & 0 & 0 & 0 & \cdots & -1 & \cdots & 1 & a_{in}-a_{ij} & & a_{in}-a_{ij} \\
1 & 1 & \cdots & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & & 1 \\
\hline
-d_1 & -d_2 & \cdots & 0 & \cdots & -d_m & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & -a_{ij} & & f-a_{ij}
\end{array}
\quad (3,7)
$$

where $d_s = a_{ij} - a_{sj}$ and $b_{st} = a_{sj} - a_{st} - a_{ij} + a_{it}$ (s = 1, 2, 3, ..., m, s ≠ i and t = 1, 2, 3, ..., n, t ≠ j).

(3) All elements, $a_{it} - a_{ij} = 0$, (t - 1, 2, 3, ..., n, t ≠ j) in the last column of the matrix (3, 7) must be non-negative except the jth element, a_{ij}, and the last one, f-a_{ij}, because a_{ij} is the minimum payoff in the ith row of the matrix game A = $[a_{ij}]$ (m, n). It is clear that we have a baisc feasible solution $x_i = 1$, $V = a_{ij}$ and $u_t = a_{it} - a_{ij}$ (t = 1, 2, 3, ..., n, t ≠ j), to start the simplex algorithm in finding the optimal solution of the matrix game.

Example 3. Find the game value and the optimal solutions (strategies) for each player as in example 2, where the matrix game A = $[a_{ij}]_{(3,4)}$ is as below:

Player C

$$\text{Player R}\quad \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ 2 & 3 & 4 & -1 \\ -2 & 5 & -3 & -1 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

From (3, 1), the matrix game can be converted into a problem of linear programming as follows:

Maximize $f = V$

Subject to

$$\begin{aligned} -2x_1 + 2x_2 - 4x_3 + V &\leqslant 0 \\ -3x_1 - 5x_2 - x_3 + V &\leqslant 0 \\ -4x_1 + 3x_2 - 3x_3 + V &\leqslant 0 \\ x_1 + x_2 - 2x_3 + V &\leqslant 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0$ and V unrestricted in sign.

After introducing 5 non-negative variables u_1, u_2, u_3, u_4, u_5, to this problem, we obtain

Maximize $f = V$

Subject to

$$\begin{aligned} -2x_1 + 2x_2 - 4x_3 + V + u_1 &= 0 \\ -3x_1 - 5x_2 - x_3 + V + u_2 &= 0 \\ -4x_1 + 3x_2 - 3x_3 + V + u_3 &= 0 \\ x_1 + x_2 - 2x_3 + \quad + u_4 &= 0 \\ x_1 + x_2 + x_3 + u_5 &= 1 \end{aligned}$$

and from (3, 6), we have

$$\begin{pmatrix} x_1 & x_2 & x_3 & V & u_1 & u_2 & u_3 & u_4 & u_5 & \\ -2 & 2 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -5 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & 3 & -3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & f \end{pmatrix}$$

matrix 1

From example 1, player R can get his expected payoff at least $a_{32} = 1$, and it is the minimum payoff in the third row of the game matrix A; therefore the non-basic variable V must be made basic instead of being the basic variable u_2 and the basic variable u_5 made non-basic instead of being the non-basic variable x_3. According to the matrix (3, 7), the matrix 1 can be adjusted as follows:

x_1	x_2	x_3	V	u_1	u_2	u_3	u_4	u_5		
4	10	0	0	1	-1	0	0	3	3	3/10
-2	-4	0	1	0	1	0	0	1	1	
1	10	0	0	0	-1	1	0	2	2	1/5
5	(7)	0	0	0	-1	0	1	1	1	1/7←
1	1	1	0	0	0	0	0	1	1	
2	4↑	0	0	0	-1	0	0	-1	f-1	

matrix 2

According to the simplex method, as the non-basic variable x_2 becomes basic instead of being the basic variable u_4 and by using elementary matrix row operations, we have

x_1	x_2	x_3	V	u_1	u_2	u_3	u_4	u_5	
$\frac{-22}{7}$	0	0	0	1	$\frac{3}{7}$	0	$\frac{-10}{7}$	$\frac{11}{7}$	$\frac{11}{7}$
$\frac{6}{7}$	0	0	1	0	$\frac{3}{7}$	0	$\frac{4}{7}$	$\frac{11}{7}$	$\frac{11}{7}$
$\frac{-43}{7}$	0	0	0	0	$\frac{3}{7}$	1	$\frac{-10}{7}$	$\frac{4}{7}$	$\frac{4}{7}$
$\frac{5}{7}$	1	0	0	0	$\frac{-1}{7}$	0	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
$\frac{2}{7}$	0	1	0	0	$\frac{1}{7}$	0	$\frac{-1}{7}$	$\frac{6}{7}$	$\frac{6}{7}$
$\frac{-6}{7}$	0	0	0	0	$\frac{-3}{7}$	0	$\frac{-4}{7}$	$\frac{-11}{7}$	$f - \frac{11}{7}$

matrix 3

Matrix 3 is terminal because all its elements at the bottom row are non-positive except the u_5 column. It is clear thar the optimal solution will be $f-\frac{11}{7} = 0$ or $f = \frac{11}{7}$ and the basic feasible solution $x_2 = \frac{1}{7}$, $x_3 = \frac{6}{7}$ and $V = \frac{11}{7}$, and player R therefore will use the optimal mixed strategy with probability vector $X^T = (x_1\ x_2\ x_3) = (0\ \frac{1}{7}\ \frac{6}{7})$ and get the expected maximum payoff $V = \frac{11}{7}$. Due to the duality theorem, player C should be expected to use the optimal mixed strategy with probability vector $Y^T = (y_1\ y_2\ y_3\ y_4) = -(0\ \frac{-3}{7}\ 0\ \frac{-4}{7}) = (0\ \frac{3}{7}\ 0\ \frac{4}{7})$ and the game value V is $\frac{11}{7}$.

Example 4. Find the optimal strategies for each player, the value of the matrix game $A = [a_{ij}]_{(3,4)}$ being as follows:

Player C

Player R

	y_1	y_2	y_3	y_4	
x_1	0	2	−2	0	−2
x_2	−3	0	4	1	−3
x_3	3	−4	0	−2	−4
	3	2	4	1	

From (1, 1), we have $\max_1 \min_3 a_{13} = -2$ and $\min_4 \max_2 a_{24} = 1$ but from (1, 6), the game value V is in the interval $-2 \leqslant V \leqslant 1$.

From (3, 1), we have th problem of linear programming as this:

Maximize $f = V$

Subject to

$$
\begin{aligned}
3x_2 - 3x_3 + V &\leqslant 0 \\
-2x_1 + 4x_3 + V &\leqslant 0 \\
2x_1 - 4x_2 + V &\leqslant 0 \\
-x_2 + 2x_3 + V &\leqslant 0 \\
x_1 + x_2 + x_3 &= 1
\end{aligned}
$$

$x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$, and V unrestricted in sign

After introducing 5 non-negative vriables, u_1, u_2, u_3, u_4, u_5, to this problem, we are to solve

Maximize $f = V$

Subject to

$$
\begin{aligned}
3x_2 - 3x_3 + V + u_1 &= 0 \\
-2x_1 + 4x_3 + V + u_2 &= 0 \\
2x_1 - 4x_2 + V + u_3 &= 0 \\
-2x_2 + 2x_3 + V + u_4 &= 0 \\
x_1 + x_2 + x_3 + u_5 &= 1
\end{aligned}
$$

and from (3, 6), the simplex matrix can be obtained as follows:

x_1	x_2	x_3	V	u_1	u_2	u_3	u_4	u_5	
0	3	−3	1	1	0	0	0	0	0
−2	0	4	1	0	1	0	0	0	0
2	−4	0	1	0	0	1	0	0	0
0	−1	2	1	0	0	0	1	0	0
1	1	1	0	0	0	0	0	1	1
0	0	0	1	0	0	0	0	0	f

matrix.1

Player R can get his expected payoff at least $a_{13} = -2$ as indicated above, and thus, the non-basic variable V must become basic in stead of being u_3, and the basic variable u_5 must become non-basic instead of being x_1. According to the matrix (3, 7), we have

x_1	x_2	x_3	V	u_1	u_2	u_3	u_4	u_5		
0	(9)	-1	0	1	0	-1	0	2	2	2/9 ←
0	8	8	0	0	1	-1	0	4	4	1/2
0	-6	-2	1	0	0	1	0	-2	-2	
0	5	4	0	0	0	-1	1	2	2	2/5
1	1	1	0	0	0	0	0	1	1	1
0	6 ↑	2	0	0	0	-1	0	2	f + 2	

matrix 2

From the simplex algorithm, we obtain the following sequence of matrices:

x_1	x_2	x_3	V	u_1	u_2	u_3	u_4	u_5		
0	1	$\frac{-1}{9}$	0	$\frac{1}{9}$	0	$\frac{-1}{9}$	0	$\frac{2}{9}$	$\frac{2}{9}$	
0	0	$\frac{80}{9}$	0	$\frac{-8}{9}$	1	$\frac{-1}{9}$	0	$\frac{20}{9}$	$\frac{20}{9}$	1/4
0	0	$\frac{-8}{3}$	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	$\frac{-2}{3}$	$\frac{-2}{3}$	
0	0	($\frac{41}{9}$)	0	$\frac{-5}{9}$	0	$\frac{-4}{9}$	1	$\frac{8}{9}$	$\frac{8}{9}$	8/41 ←
1	0	$\frac{10}{9}$	0	$\frac{-1}{9}$	0	$\frac{1}{9}$	0	$\frac{7}{9}$	$\frac{7}{9}$	7/10
0	0	$\frac{8}{3}$ ↑	0	$\frac{-2}{3}$	0	$\frac{-1}{3}$	0	$\frac{2}{3}$	$f + \frac{2}{3}$	

matrix 3

x_1	x_2	x_3	V	u_1	u_2	u_3	u_4	u_5	
0	1	0	0	$\frac{4}{41}$	0	$\frac{-5}{41}$	$\frac{1}{41}$	$\frac{10}{41}$	$\frac{10}{41}$
0	0	0	0	$\frac{8}{41}$	1	$\frac{31}{41}$	$\frac{-80}{41}$	$\frac{20}{41}$	$\frac{20}{41}$
0	0	0	1	$\frac{14}{41}$	0	$\frac{3}{41}$	$\frac{24}{41}$	$\frac{-6}{41}$	$\frac{-6}{41}$
0	0	1	0	$\frac{-5}{41}$	0	$\frac{-4}{41}$	$\frac{9}{41}$	$\frac{8}{41}$	$\frac{8}{41}$
1	0	0	0	$\frac{1}{41}$	0	$\frac{9}{41}$	$\frac{-10}{41}$	$\frac{23}{41}$	$\frac{23}{41}$
0	0	0	0	$\frac{-14}{41}$	0	$\frac{-3}{41}$	$\frac{-24}{41}$	$\frac{6}{41}$	$f + \frac{6}{41}$

matrix 4

Matrix 4 is terminal because all its elements at the bottom are nonpositive except the u_5 column. It is clear that the optimal solution $f + \frac{6}{41} = 0$ or $f = \frac{-6}{41}$ and the basic feasible solution $x_1 = \frac{-23}{41}$, $x_2 = \frac{-10}{41}$, $x_3 = \frac{-8}{41}$ and $V = \frac{-6}{41}$. Player R will therefore use the the mixed optimal strategy with probability vector $X^T = (x_1\ x_2\ x_3) = (\frac{23}{41}\ \frac{10}{41}\ \frac{8}{41})$ and get the maximum expected payoff $V = \frac{-6}{41}$ or lose the minimum expected payoff $\frac{6}{41}$ to player C. Due to the duality theorem, player C will be ecpected to use the optimal mixed strategy with probability vector $Y^T = -(\frac{-14}{41}\ 0\ \frac{-3}{41}\ \frac{-24}{41}) = (\frac{14}{41}\ 0\ \frac{3}{41}\ \frac{24}{41})$ and the game value V is $\frac{-6}{41}$

4. REMARKS

(1) If the value V of a matrix game is zero or negative, its problems of linear programming (2, 5) and (2, 6) are not defined. In general, if a matrix game has at lèast one row with all positive elements, its value V is positive (7. P. 275). Hence, a positive constant can be added to each element of the matrix game to change it into positive value while each player strategy is not affected by this change though its value V will certainly be (2, P. 643, and 5, P. 408). From the relationship (1, 6), the value of a matrix game $A = [a_{in}]_{(m,n)}$ is in the interval $\max_i \min_j a_{ij} \leq V \leq \min_j \max_i a_{ij}$ and it is clear that the interval of V can be modified by the transformation through adding a positive constant, k, to each one of this interval, $\max_i \min_j a_{ij} + k \leq V + k \leq \min \max a_{ij} + k$ to make sure $V + k$ being in the positive interval.

Example 5. From example 4, the game value V is in the interval $-2 \leq V \leq 1$ and it can be changed into $-2 + 3 \leq V + 3 \leq 1 + 3$ by adding a positive constant 3 to each. Hence, the matrix game $A = [a_{ij}]_{(3,4)}$ can be changed into the matrix game $A_1 = [a_{ij} + 3]_{(3,4)}$.

or

Player C

		y_1	y_2	y_3	y_4
Player R	x_1	3	5	1	3
	x_2	0	3	7	4
	x_3	6	-1	3	1

Let $V' = V + 3$ be the value of the matrix game A_1, V' is in the interval $1 \leq V' \leq 4$. From (2, 6), the problem of linear programming can be obtained as follows:

$$\begin{aligned} \text{Maximize} \quad \frac{1}{V'} &= q_1 + q_2 + q_3 + q_4 \\ \text{Subject to} \quad & 3q_1 + 5q_2 + q_3 + 3q_4 \leqslant 1 \\ & 3q_2 + 7q_3 + 4q_4 \leqslant 1 \\ & 6q_1 - q_2 + 3q_3 + q_4 \leqslant 1 \\ & q_1 \geqslant 0, q_2 \geqslant 0, q_3 \geqslant 0, \text{and } q_4 \geqslant 0. \end{aligned}$$

By introducing 3 non-negative variables $v_1 \leqslant 0$, $v_2 \geqslant 0$, and $v_3 \geqslant 0$, the following simplex matrix can be obtained

$$\begin{array}{cccc|ccc|c} q_1 & q_2 & q_3 & q_4 & v_1 & v_2 & v_3 & \\ 3 & 5 & 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 3 & 7 & \textcircled{4} & 0 & 1 & 0 & 1 \\ 6 & -1 & 3 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & \frac{1}{V'} \end{array}$$

matrix 1

The non-basic variable q_4 maust first be made basic instead of being the basic variable v_2 because player C wants to minimize his expected maximum payoff to player R at most 4. Hence, the pivot is 4 by using elementary matrix row operations, we have

$$\begin{array}{cccc|ccc|cl} q_1 & q_2 & q_3 & q_4 & v_1 & v_2 & v_3 & & \\ \textcircled{3} & \frac{11}{4} & \frac{-17}{4} & 0 & 1 & \frac{-3}{4} & 0 & \frac{1}{4} & 1/12 \leftarrow \\ 0 & \frac{3}{4} & \frac{7}{4} & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \\ 6 & \frac{-7}{4} & \frac{5}{4} & 0 & 0 & \frac{-1}{4} & 1 & \frac{3}{4} & 1/8 \\ \hline 1 & \frac{1}{4} & \frac{-3}{4} & 0 & 0 & \frac{-1}{4} & 0 & \frac{1}{V'}-\frac{1}{4} & \end{array}$$

matrix 2

From the simplex algorithm, we obtain the following sequence of matrices:

$$\begin{array}{cccc|ccc|c} q_1 & q_2 & q_3 & q_4 & v_1 & v_2 & v_3 & \\ 1 & \frac{11}{12} & \frac{-17}{12} & 0 & \frac{1}{3} & \frac{-1}{4} & 0 & \frac{1}{12} \\ 0 & \frac{-3}{4} & \frac{7}{4} & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{-29}{4} & \textcircled{\frac{39}{4}} & 0 & -2 & \frac{5}{4} & 1 & \frac{1}{4} \\ \hline 0 & \frac{-8}{12} & \frac{8}{12}\uparrow & 0 & \frac{-1}{3} & 0 & 0 & \frac{1}{V'}-\frac{1}{3} \end{array} \begin{array}{l} \\ \\ 1/7 \\ 1/39 \leftarrow \\ \\ \end{array}$$

matrix 3

$$\begin{array}{cccc|ccc|c} q_1 & q_2 & q_3 & q_4 & v_1 & v_2 & v_3 & \\ 1 & \frac{100}{117} & 0 & 0 & \frac{5}{117} & \frac{28}{117} & \frac{17}{117} & \frac{14}{117} \\ 0 & \frac{55}{39} & 0 & 1 & \frac{14}{39} & \frac{1}{39} & \frac{-7}{39} & \frac{8}{39} \\ 0 & \frac{-29}{39} & 1 & 0 & \frac{-8}{39} & \frac{5}{39} & \frac{4}{39} & \frac{1}{39} \\ \hline 0 & \frac{-20}{117} & 0 & 0 & \frac{-23}{117} & \frac{-10}{117} & \frac{-8}{117} & \frac{1}{V'}-\frac{41}{117} \end{array}$$

matrix 4

Matrix 4 is terminal and the optimal solution $\frac{1}{V'} - \frac{41}{117} = 0$ or $V' = \frac{117}{41}$ and the vector $Q^T = (q_1\ q_2\ q_3\ q_4) = (\ 14\ \ 0\ \frac{1}{39}\ \ \frac{8}{39}\ )$ and due to the duality theorem the vector $P^T = (p_1\ p_2\ p_3) = -(\frac{-23}{117}\ \ \frac{-10}{117}\ \ \frac{-8}{117}) = (\frac{23}{117}\ \ \frac{10}{117}\ \ \frac{8}{117})$. For the matrix game A, player C has the mixed optimal stategy with the probability vector $Y^T = (y_1\ y_2\ y_3\ y_4) = V'(q_1\ q_2\ q_3\ q_4) = (\frac{14}{41}\ \ 0\ \ \frac{3}{41}\ \ \frac{24}{41})$; player R has the mixed optimal startegy with probability vector $X^T = (x_1\ x_2\ x_3) = V'(p_1\ p_2\ p_3) = (\frac{23}{41}\ \ \frac{10}{41}\ \ \frac{8}{41})$; and the game value V is $V' - 3 = \frac{-6}{41}$ which are all being the same solution as indicated in example 4 above.

(2) By using this result to find the optimal solution for a matrix game, the last column and the last row of the simplex matrix (3, 7) can be omitted to simplify the matrix form. This can be done because the last two columns are identical and the last row and the pivot row of V have opposite signs. From example 4, through the last column and the last row are omitted, the algorithmic steps and the optimal solution are not affected. It can be clearly seen that the problems of linear programming (3, 1) and (3, 3) of a matrix game are more efficient than the problems of linear programming (2, 5) and (2, 6).

5. REFERENCE

1. Mangasarian, O. L., and Stone, H. "Two-person Nonzero-sum Games and Quadratic Programming," Journal of Mathematical Analysis and Applications 9, PP348-355, 1964.
2. Drapper, J. E., and Klingman, J. S. "Mathematical Analysis: Business and Economic Applications," Harper and Row Publishers, Inc., N. Y. 2nd. Edi. PP 691, 1972.
3. Dresher, R. M. "Game of Strategy: Theory and Applications," Pretice Hall, Inc., Englewood Cliffs N. J. PP 186, 1963.
4. Charnes, A., and Cooper, W. W. "Management Models and Industrial Applications of Linear Programming," Vol. 2, Wiley, N. Y., 1961.
5. Ruce, R. D., and Raiffa, H. "Game and Decisions," John Wiley and Sons, Inc., N. Y., PP 509 1957.
6. Campbell, H. G. "An Introduction to Matrices, Vectors and Linear programming," Prentice Hall, Inc., Englewood Cliffs, N. J., 2nd. Edi., PP 316, 1977.
7. Thie, P. R. "An Introduction to Linear Programming anf Theory," John Wiley and Sons, Inc., PP 335, 1979.