where λ is the step length and d is a feasible search direction. If the current basic feasible solution, ycur, equals to $(x_6, x_3, x_2)^T = (10/3, 5/6, 8/3)^T$, can you find the set of all feasible search directions, and point out which direction will lead ycur to the optimum ? What is the value of λ for reaching the optimum.

II. Continue from problem I.

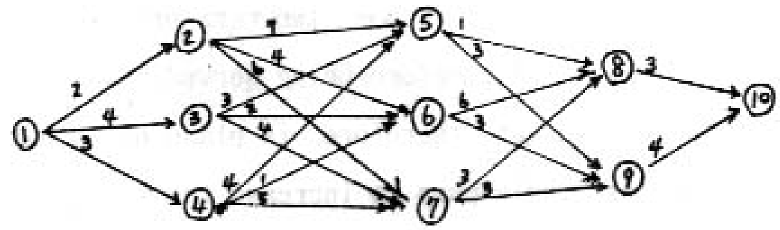
(a) Write down the dual problem to (P).

(b) State its complementary slackness conditions.

(c) Prove the following theorem.

In a primal-dual pair of problem (P) and its dual problem, if either the primal or the dual problem has an optimal feasible solution, then the other does also, and the two optimal objective values are equal.

III. The following figure represents a network of ten different cities. Every arc with an arrow sign denotes the one-way route and a distance in numbered above it. Based on the dynamic programming formulation, you are asked to answer the following questions.



(a) Find the shoutest path from node 1 to node 10 in the network. Also, find the shortest path from node 3 to 3 to node 10.

(b) Write down the stages, states, decisions, returns and the recursive equation when consider a dynamic programming approach.

(c) Write down the algorithm to find the longest path in the network. Prove your algorithm has finite interations to reach the objective, and it is satis fied with the priciple of optimality.

IV. Suppose that all car owners fill up when their tanks are exactly half full. At the present time, an average of 7.5 customers per hour arrive at a single-pump gas station. It takes an average of 4 minutes to serve a car. Assume that interarrival times and service times are both exponential.

(a) For the present situation, find the probability there are at most three customers in the system.

(b) Compute the average length, L, and the average system time, W.

(c) Suppose there are at most two cars are allowed in the waiting line before being served. The gas station manager, therefore, is planning to put an similiar single-pump to increase the total service rate, or improve other performance measures. According to the following three proposals, which proposal will you suggest the manager to take. Please use Mathematical arguments to verify your answer.

(1) Put the new pump besides the old one to share the same waiting line;

(2) Put the new pump after the old one without waiting room in between;

(3) Put the new one in parallel to the old one and give its own waiting room of which the capacity is one before service.

國立政治大學應用數學研究所碩士班
研究生入學考試試題

八十三學年度

微積分

#1. (10%) Given $\epsilon = 1$, find $\delta > 0$ with the property that

$$\left| \frac{x^3-8}{x-2} - 7 \right| < \epsilon \quad \text{whenever } 0 < |x-1| < \delta .$$

#2.(10%) Find the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left\{ \left(\frac{k}{n}\right)^{\frac{1}{2}} - \left(\frac{k}{n}\right)^{\frac{3}{2}} \right\} \frac{1}{n}$$

#3. (10%) Find the solution of the differential equation that satisfies the given condition

$$y\,(1-x^2)\,y' + x\,(y^2+1) = 0\,, \qquad y(\sqrt{2}) = 2\,.$$

#4. (10%) Consider

$$f(x, y) = (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) \quad , \text{ if } (x, y) \neq (0, 0),$$

$$= 0 \quad , \text{ if } (x, y) = (0, 0).$$

(1) find $f_x(0,0)$ and $f_y(0,0)$

(2) show that f is differentiable at (0,0),

(3) show that f_x and f_y are not continuous at (0,0).

#5. (10%) (1) Evaluate the line integral

$$\int_C xy\,dx + x\,dy$$

over the simple closed curve C which is formed by the line segment L_1 from (0,0) to (0,2), line segment L_2 from (0,2) to (1,1) and line segment L_3 from (1,1) to (0,0).

(2) Use Green's Theorem to find the above integral

#6. (10%) Show that the surface area of a sphere of radius R is $4\pi R^2$.

#7. (10%) If a function g satisfies the following properties
:

(1) g is differentiable on [0,1].

(2) a number L exists with $|g'(x)| \leq L < 1$ for all $x \in [0,1]$.

(3) $0 \leq g(x) \leq 1$ for all $x \in [0,1]$.

Show that

a) g has a unique fixed point p in [0,1], that is, $g(p)=p$.

b) if $x_0 \in [0,1]$ and $x_n = g(x_{n-1})$ for each $n \geq 1$, then $\lim_{n\to\infty} x_n = p$.

#8. (10%) Suppose $\sum_{n=1}^{\infty} A_n$ is a convergent series of positive term and that $\{B_n\}$ is a sequence with $\lim_{n\to\infty} B_n = B > 0$. Show that $\sum_{n=1}^{\infty} A_n B_n$ AnBn converges.

#9. (10%) Find the area of the region lying inside the graphs of both polar equations $r=1$ and $r=2\sin\theta$.

#10. (10%) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n^p}$$

in the following cases:

(1) $p=1$;

(2) $p=2$;

(3) $p=0.5$.

線性代數

1. Let $(V, \langle , \rangle)$ be an n-dimensional inner-product space over $F = \mathbb{R}$ or $\mathbb{C}$, $g: V \to F$ a linear functional.

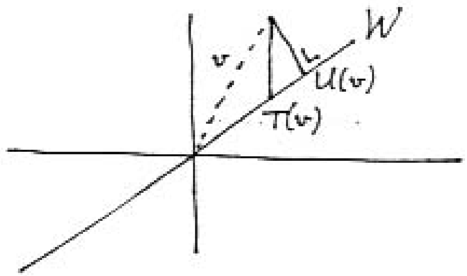
(a) Show that $\exists$ a unique $y \in V$ such that $\forall x \in V$
$g(x) = \langle x, y \rangle$. (RIESZ's Lemma)

(b) For a linear operator $T: V \to V$, define the adjoint operator T^* and the transpose operator T^t and show that for any orthonormal basis $\beta \subseteq V$: $[T^*]_\beta = [T]_\beta^*$. Is $[T^*]_\beta = [T^t]_{\beta^*}$, where β^* is the dual basis of β? Why?

(c) Prove: $Im(T^*) = (Ker\, T)^\perp$. (30%)

2. (a) Let $(V, \langle , \rangle)$ be as in 1. and $W \leq V$ a subspace. Give definitions and characterizations of projections and orthogonal projections on W. Show that every orthogonal projection is self-adjoint.

(b) Show that $T, U: \mathbb{R}^2 = V \to W = \mathbb{R}.(1,1)$ defined below are a projection and an orthogonal projection on W, respectively:



$\mathbb{R}^2$

3. For each of the following operators T test T for diagonalizability and find a basis $\beta \subseteq V$ such that $[T]_\beta$ is a diagonal matrix.

(a) $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$, $f \mapsto f' + f''$ where $P_3(\mathbb{R}) = \{ f(x) \in \mathbb{R}[x] \mid \deg f \leq 3 \}$ and $(\ )'$ means the derivation.

(b) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2b - 3c \\ a + 3b + 3c \\ c \end{pmatrix}$

(30%)

4. Find the Jordan canonical form and a Jordan canonical basis of the following matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$$

機率統計

1. For a nonnegative integer-valued random variable N, show that (20%)

$$\sum_{i=0}^{\infty} i\, P\{N > i\} = \frac{1}{2}\left(E[N^2] - E[N]\right)$$

2. A point is chosen at random on a line segment of langth 1. Find the probability that the ratio of the shorter to the longer segment is less then 1/4. (15%)

3. Suppose that the joint density of X and Y is given by(15%)

$$f(x,y) = \frac{1}{y} e^{-y}, \quad 0<x<y, \ 0<y<\infty$$

Compute $E[X^3 \mid Y=y]$

4. Let $f(x;\theta) = \frac{1}{\theta} X^{\frac{1-\theta}{\theta}}$ $0<x<1$, $0<\theta<\infty$. (20%)

(1) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

(2) Is $\hat{\theta}$ an unbiased estimator of θ? why?

5. Let X have the binomial distribution b(n, p). Find a uniformly most powerful critical region for testing $H_0: p=p_0$ against $H_1: p<p_0$ (15%)

6. Suppose $Y_i = \theta_1 + \varepsilon_i$, $i=1,2,\ldots,n_1$ and $Y_i = \theta_2 + \varepsilon_i$, $i=n_1+1, n_1+2, \ldots n_1+n_2$; where $\varepsilon_1, \ldots, \varepsilon_{n_1}, \varepsilon_{n_1+1}, \ldots \varepsilon_{n_1+n_2}$ are independent $N(0, \sigma^2)$ variables. Find the least squares estimator of $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$. (15%)

作業研究

I. Consider the following linear programming problem (P). Variables x5, x6, x7 are the slack variables corresponding to the various inequalities, the basis B1 corresponding to the basic vector (x_1, x_3, x_2) is optimal to the problem.

$$\begin{aligned} \text{minimize} \quad & z(x) = -2x_1 - 4x_2 - x_3 - x_4 \\ \text{subject to} \quad & x_1 + 3x_2 + x_4 \le 8 \quad (1) \\ & 2x_1 + x_2 \le 6 \quad (2) \\ & x_2 + 4x_3 + x_4 \le 6 \quad (3) \\ & x_j \ge 0 \quad \text{for all } j \end{aligned}$$

(a) If the inverse matrix, $B_1^{-1} = \begin{bmatrix} -1/5 & 3/5 & 0 \\ -1/10 & 1/20 & 1/4 \\ 2/5 & -1/5 & 0 \end{bmatrix}$, find the optimum primal and simplex multipliers associated with B1.

(b) If the right hand side of constraints (1), (2) and(3) represents the available amount of materials 1, 2 and 3, which one should be picked, when the availability of only one of the raw materials can be marginally increased? Why?

(c) For what range of values of b1, the right hand side of constraint (1), does the basis B1 remain optimal?

(d) According to the algorithm of the simplex method, it travels vertices in the feasible region along the path to an optimal soluion. In other words, the current basic feasible solution, or a vertex denoted as ycur, will visit yadj, one of its adjacent vertices, by the following way,

$$y_{adj} = y_{cur} + \lambda d$$