考試科目	線性代數	所別	應用數學研究所	考試時間	4月23日 星期日 上午第2節

注意：沒有計算過程，只寫答案者，不給分。

1. (20%) (a) If A is an $m\times n$ matrix and if there is an $n\times m$ matrix B, such that $AB=I_m$, show that the equation $Ax=y$ is always solvable.

(b) Using the fact that

$$\begin{bmatrix} -1 & 2 & 0 \\ 4 & -9 & 1 \end{bmatrix}\begin{bmatrix} -9 & -2 \\ -4 & -1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

find solutions to the equations

$$\begin{aligned} -x+2y \quad &= \alpha \\ 4x-9y+z &= \beta \end{aligned}$$

2. (10%) If $A\neq 0$ is a symmetric matrix and $B\neq 0$ is a skew-symmetric matrix in the space of $n\times n$ matrices, show that A and B are linearly independent.

3. (30%) Let P_2 be the space of all polynomials (with real coefficients) of degree less than or equal to 2, and let T be the linear operator on the space P_2 defined by the equation:

$$T(a+bx+cx^2)=(4a+6c)(x^2-1)+2(b-a-c)x$$

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(a) Find a basis for the range of T.
(b) Find a basis for $\ker T$.

4. (20%) (a) Let A be an $m \times n$ matrix. Show that for any x in $\mathbb{R}^n$ and any y in $\mathbb{R}^m$,

$$(Ax, y) = (x, A^T y)$$

where on the left we have the usual inner product in $\mathbb{R}^m$ and on the right we have the usual inner product in $\mathbb{R}^n$.

(b) Assume that A is a symmetric $n \times n$ matrix. Assume that x and y are vectors in $\mathbb{R}^n$ with the property that $Ax = 2x$ and $Ay = 7y$. Prove that x and y are orthogonal.

5. (20%) Find the Jordan normal form of the matrix

$$A = \begin{pmatrix} -2 & 1 & 3 & -1 \\ 3 & 0 & -2 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & -3 & 0 \end{pmatrix}.$$

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I. Consider a linear programming problem (LP) shown below.(25%)

$$
\begin{array}{llcrcrl}
\max\ z & = & -3x_1 & + & 6x_2 & & \\
\text{s.t.} & & 5x_1 & + & 7x_2 & \le 35 \\
 & & -x_1 & + & 2x_2 & \le 2 \\
 & & x_1, & & x_2 \ge 0 & &
\end{array}
$$

(a) Does this linear programming problem have more than one basic feasible solution that is optimal? why?

(b) If the answer in (a) is yes, how will you construct the set for all optimal solutions?

(c) Prove the set of optimal solutions to a linear programming problem is a convex set.
(Hint: A subset X of R^n is called a convex set if for any two points $x^1, x^2 \in X$, the line segment joining them is also in X.)

II. TeleCom produces television sets at two plants both of which can produce up to 30 sets per week. Television sets are shipped to three customers whose demands are 20 sets per week. The profit earned per TV set depends on the site where the set was produced and the customer who purchases the TV set. They are shown in Table II.(25%)

From	To		
	Customer 1	Customer 2	Customer 3
Plant 1	\$4000	\$2000	\$4000
Plant 2	\$12000	\$8000	\$4000

Table II

(a) Formulate a linear programming problem that can be used to maximize the TeleCom's profits.

(b) Can you find a basic feasible solution for this problem? Explain briefly the method you use.

(c) Find an optimal solution to the problem.

(d) Determine the range of values on the profit associated with the plant 1 and the customer 3 for which the current basis remains optimal.

(e) If the production in plant 1 and the demand of Customer 3 are both increased by 5, what is the new optimal solution to the TeleCom problem.

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III. Garry is determining how many of three types of objects should be brought in the baggage for his business traveling. The weight and benefit of each of the items are given in Table III. If the baggage can carry up to 33 lb of items 1-3, which items should be taken for maximizing the benefit in his travel?(25%)

	WEIGHT	BENEFIT
Item 1	3 lb	12
Item 2	5 lb	25
Item 3	7 lb	35

Table III

(a) Formulate this problem in a Mathematical form.

(b) Explain with clear logic that the method or algorithm you use to solve this problem.

(c) Solve this problem with your method in (b), and write down the optimal solutions

IV. Consider a problem of Markov chain with transition matrix P given below.(25%)

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

(a) The probability vector, π, is called stationary if

$$\pi = \pi P.$$

Find the the stationary probabilities, π.

(b) If all states in a chain are recurrent, aperiodic, and communicate with each other, the chain is said to be ergodic. Thus, $\pi = (\pi_1, \pi_2)$ is called the steady-state probabilities if the chain is ergodic and $\pi = \pi P$. Suppose π is the steady-state probabilities and that with probability π_i, the Markov chain begins in state i. What is the probability that after one transition, the system will be in state i?

(c) For any value of n ($n = m + 1, m + 2, \ldots$), where m is a very large integer, what is the probability a Markov chain will be in state i after n transitions?

(d) If we let

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then we know $\pi = (1/2, 1/2)$ is a stationary probability vector. But, is π a steady state probability vector? why?

(e) A square matrix is said to be ergodic, doubly stochastic if its entries are all nonnegative and the entries in each row and each column sum to 1. For doubly stochastic matrix, show that all states have the steady-state probability.

考試科目	機率與統計	所別	應用數學所	考試時間	4月23日 星期日 上下午第 節

1. Consider two random variables X and Y having a joint probability density function (18%)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}xy & \text{if } 0 \le y \le x \text{ and } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Are X and Y independent?

(ii) Find the probability $P[1 \le X+Y \le 3]$.

(iii) Find the conditional probability $P[X<1 \mid Y<1]$.

2. (16%) Consider n independent trials each resulting in any one of r possible outcomes with probabilities $P_1, P_2, \cdots, P_r$. Find the expected number of outcomes that never occur in any of the trials. Show that this expected number is minimized when $P_i = 1/r$, $i=1,\cdots,r$. (among all possible vectors $P_1,\cdots,P_r$.)

3. (16%) A system will function as long as at least one of three components functions. When all three components are functioning, the distribution of the life of each is exponential with parameter $\frac{1}{3}\lambda$. When only two are functioning, the distribution of the life of each of the two is exponential with parameter $\frac{1}{2}\lambda$; and when only one is functioning, the distr. of its life is exponential with parameter λ.

(i) What is the distribution of the lifetime of the system?

(ii) Suppose now that only one component (of the three components) is used at a time and it is replaced when it fails. What is the distribution of the lifetime of such a system?

4. Let X_i be a random variable distributed uniformly on $[\mu-\sqrt{3}\sigma, \mu+\sqrt{3}\sigma]$
(20%) where $-\infty<\mu<\infty$ and $\sigma>0$, and X_i, X_j are independant for $i \neq j$.

(i) Find the estimator of (μ, σ) by the method of moments.

(ii) Find the estimator of (μ, σ) by the maximal likelihood the method

(iii) Compare these two estimators (e.g. unbiasedness, variance)

5. (i) Let $X_1, \cdots, X_n$ be a random sample from the Poisson density
(20%)

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x=0,1,\cdots$$

Find the UMVUE of $e^{-\lambda}$.

(ii) Let $T = \frac{1}{n}\sum_{i=1}^{n} I_{\{0\}}(X_i)$, where $I_{\{0\}}(X_i) = \begin{cases} 1 & \text{if } X_i = 0 \\ 0 & \text{o.w.} \end{cases}$

Show that is also unbiased for $e^{-\lambda}$ and it is worse than the UMVUE.

6. Let X be a discrete random variable with density given by
(10%)

x	0	1	2	3
$f(x;0)$	0.05	0.05	0.10	0.80
$f(x;1)$	0.05	0.15	0.50	0.30

Find the most power test for testing $H_0: \theta=0$ against $H_1: \theta=1$ of size 0.05.

共十題，每題十分.

1. Determine the least value m of n for which it is true that

$$\frac{n^2+n+1}{3n^2+1} - \frac{1}{3} < \epsilon \quad \text{where } \epsilon \text{ is a given positive number.}$$

If $\epsilon = 0.01$; $m = ?$

2. Find the following limits:

(i) $\lim\limits_{n\to\infty} (n^2)^{\frac{1}{n}} = ?$ (ii) $\lim\limits_{n\to\infty} [\, 2^{-\frac{1}{n}} + 3^{-\frac{1}{n}} \,]^n = ?$

3. Let $S(x) = x^2 + x^2(1-x^2) + x^2(1-x^2)^2 + \dots + x^2(1-x^2)^n + \dots\dots$; when $0 < |x| < 1$;
and $S(0) = 0$.
Is $S(x)$ continuous at $x = 0$?

4. Let P and Q be any two points on the parabolas $y^2 = 4x$; $y^2 = 2x - 6$.
Find the minimum distance between P and Q .

5. Evaluate the line integral $\int_C x^4\,dy - y^4\,dx$, where C is the semi-circumference of the circle $(x-1)^2 + y^2 = 1$ which runs from the origin $(0, 0)$ to $(2, 0)$ in the first quadrant.

6. (i) Find the Taylor expansion of $f(x) = \log_e(1+x)$, $-1 < x < 1$.
(ii) Find the approximate value of $\log_e(1.01)$ with the error < 0.00001 .

7. Find the following limit

$$\lim_{n\to\infty} \int_0^x \frac{t^{4n}}{1+t^{n}}\,dt = ? \qquad \text{where } |x| < 1 ,\ n \in \mathbb{N} .$$

8. Using the known fact $\displaystyle\int_0^\infty \frac{\sin b x}{x}\,dx = \frac{\pi}{2}$ if $b > 0$, find $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = ?$.

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9. If

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{(x^2 + y^2)}, & \text{when } (x, y) \neq (0, 0), \\ 0 , & \text{when } (x, y) = (0, 0). \end{cases}$$

find the second order partial derivatives $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

10. Find the volume of the solid whose base is the rectangle $R = \{ (x, y) \mid 0 \le x \le 1,\ 0 \le y \le x \}$ and whose height at (x,y) is given by $f(x, y) = x - y$.