考試科目	線性代數	所別	應用數學系	考試時間	4月26日 星期日 上午第 2 節

space of L_A.

(4) Extend your basis for the nullspace of L_A to a basis for $\mathbb{R}^6$.

(Ⅲ) Let $M_{n\times n}(\mathbb{R})$ be the set of all $n\times n$ matrices over $\mathbb{R}$.
Let $T: M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$ be defined by
$T(A) = A^t$, the transpose of A.

(1) Verify that T is a linear transformation. Find the nullity and rank of T.

(2) Find the eigenvalues and eigenvectors of T.

(3) Can T be diagonalized?

(Ⅳ) Let A be an $n\times n$ real matrix. Show that

$$\min_{\substack{x\in\mathbb{R}^n\\ x\neq 0}} \frac{x^tAx}{x^tx} = \lambda_{min} \quad \text{and} \quad \max_{\substack{x\in\mathbb{R}^n\\ x\neq 0}} \frac{x^tAx}{x^tx} = \lambda_{max}$$

where λ_{min}, λ_{max} are the smallest, largest eigenvalue of A.

(Ⅴ) Let A be an $m\times n$ real matrix with rank of A less than m and n. Show that

(i) $\mathcal{N}(A^tA) = \mathcal{N}(A)$. (the nullspace of A^tA, A.)

(ii) $\mathcal{R}(A^tA) = \mathcal{R}(A^t)$. (the column space of A^tA, A^t)

(iii) The normal equation $A^tAx = A^tb$ is always solvable for any $b\in\mathbb{R}^m$.

(Ⅰ) 20% (Ⅱ) 25% (Ⅲ) 25% (Ⅳ) 15% (Ⅴ) 15%

備考 試題隨卷繳交

考試科目	微積分	所別	應用數學	考試時間	4月26日 星期日 上下午第 一 節

1. Show that if $f(x)$ has a limit as x approaches a, then the limit is unique. (10%)
2. Show that if f is differentiable at a, then f is continuous at a. (10%)

3. Show that if g is differentiable at x and f is differentiable at $g(x)$, then $f\circ g$ is differentiable at x and $(f\circ g)'(x) = f'(g(x))\,g'(x)$. (10%)
4. Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Show that if $f'(x)>0$ for every x in (a,b), then f is increasing on $[a,b]$. (10%)
5. Determine if $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{n+i}$ exists. (Justify your answer.) (10%)
6. Show that if f and g are integrable on $[a,b]$, then $f+g$ is also integrable on $[a,b]$ and $\int_a^b [f(x)+g(x)] = \int_a^b f(x)dx + \int_a^b g(x)dx$. (10%)
7. Suppose f is continuous on a closed interval $[a,b]$ and $G(x) = \int_a^x f(t)dt$ for every x in $[a,b]$. Show that $G(x)$ is an antiderivative of f on $[a,b]$. (10%)
8. Prove Simpson's rule using parabolic estimate. (10%)
9. Let $u = u(x,y,z)$ be a continuously differentiable function, and suppose that the equation $u(x,y,z)=0$ defines z implicitly as a differentiable function of x and y. Show that $\frac{\partial z}{\partial x} = -\frac{\partial u/\partial x}{\partial u/\partial z}$ if $\frac{\partial u}{\partial z} \neq 0$. (10%)
10. Let f and g be continuous functions on the closed and bounded region R, and $g \geq 0$ on R. Show that there exists a point (x_0, y_0) in R such that $\iint\limits_R f(x,y)\,g(x,y)\,dx\,dy = f(x_0,y_0) \iint\limits_R g(x,y)\,dx\,dy$ (10%)

考試科目	線性代數	所別	應用數學系	考試時間	4月26日 星期日 上/下午第2節

(I) True or False: Give reasons or counterexamples.

(1) Let $\mathcal{B}$ be a basis for the vector space V over F and let W be a subspace of V, then there is a basis $\mathcal{B}_1$ of W such that $\mathcal{B}_1$ is a subset of $\mathcal{B}$.

(2) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $Tx = 2x$ for some $x \in \mathbb{R}^n$, then 2 is an eigenvalue of T.

(3) Without loss of generality, we may always assume that the square matrix A in the quadratic form x^tAx is symmetric.

(4) Let $P_n(\mathbb{R})$ be the set of all polynomials of degree less than or equal to n over $\mathbb{R}$. There exists a basis for $P_n(\mathbb{R})$ over $\mathbb{R}$ consisting of $n+1$ polynomials $p_1, p_2, \cdots, p_{n+1}$ such that each p_i is of degree n.

(II) Let

$$A = \begin{bmatrix} 1 & 3 & -3 & -2 & 7 & -2 \\ -1 & -2 & 2 & 0 & -5 & -1 \\ -2 & -2 & 2 & -3 & -6 & -1 \\ 0 & 1 & -1 & -2 & 2 & -3 \\ -3 & -4 & 4 & -3 & -11 & -2 \end{bmatrix}$$

(1) Find the row echelon form of A. (The process can be skipped) 計算過程不必寫在答案卷上

(2) $A_{5\times 6}$ defines a linear transformation from $\mathbb{R}^6$ to $\mathbb{R}^5$ ($L_A(x) = Ax$). Find the nullity and rank of L_A.

(3) Find a basis for the range of L_A and a basis for the null