考試科目 Course	微積分	開課系級 Dept. & Class	應數系	日期 Date, Period	月 日 第 節	試題編號 Course No.

Definition 1: A map $\|\cdot\|$ in a vector space X is called a norm (and $(X, \|\cdot\|)$ is called a normed space), if the map $\|\cdot\| : X \to \mathbb{R}^+$ has the properties

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in \mathbb{R}, \forall x \in X$

(iii) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X.$

Definition 2: A set S in a normed space $(X, \|\cdot\|)$ is said to be open, if for every point $s \in S$ there exists a positive number $r > 0$ such that

$$B_r(s) = \{x \in X : \|x - s\| < r\} \subset S.$$

Definition 3: For the normed spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$, a bounded linear map $L : (X, \|\cdot\|) \to (Y, \|\cdot\|')$ means that L is a linear map and continuous, that is,

$$L(\alpha x_1 + \beta x_2) = \alpha L(x_1) + \beta L(x_2) \quad \forall x_1, x_2 \in X; \forall \alpha, \beta \in \mathbb{R}$$

and there exists a constant c such that

$$\|L(x)\| \leq c \|x\| \quad \forall x \in X.$$

Definition 4: Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and $S \subset (X, \|\cdot\|_X)$ is an open set in $(X, \|\cdot\|_X)$. Then the map $f : S \to (Y, \|\cdot\|_Y)$ is called differentiable in S, if for every point $s \in S$ there exists a bounded linear map $L_s : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ such that

$$f(s + h) = f(s) + L_s(h) + \varepsilon(h),$$

where $\varepsilon(h)$ satisfies $\lim_{\|h\| \to 0} \frac{\|\varepsilon(h)\|_Y}{\|h\|_X} = 0.$

V. Prove the following statements (30%)

7. For a 2×2 matrix $A = (a_{ij})_{2\times 2}$, we set $\|A\| = \sum_{i,j=1}^{2} |a_{i,j}|$, then $\|\cdot\| : M_2 \to \mathbb{R}$ is a norm, where M_2 is the collection of all 2×2 real matrices.(10%)

8. The set $M = \left\{ A \in M_2 : \left\| A - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\| < 1 \right\}$ is open in $(M_2, \|\cdot\|)$. (10%)

9. The function $f(A) = A^2$ is differentiable in M. (10%)

線性代數	開課系級 Dept, & Class	應數所	日期 Date, Period	4 月 25 日 第 2 節	試題編號 Course No

I. True or false. Prove your assertions or give counterexamples.

(i) Let $W_1, W_2 \leq V$ be two subspaces of a finite-dimensional F-vector space V, then the dimension $dim_F(W_1 + W_2) = dim_F W_1 + dim_F W_2$.

(ii) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation, and $\lambda_1, \lambda_2 \in spec(T)$ two eigenvalues of T with eigenspaces $E_{\lambda_i}, i = 1, 2$, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

(iii)Let $T : F^n \rightarrow F^n$ be a linear transformation with $spec(T) = \{\lambda_1, \lambda_2\}, \lambda_1 \neq \lambda_2$ and $dim_F E_{\lambda_1} = n - 1$, then T is diagonalizable over F.

(iv) Let $P_n(\mathbb{R}) = \{f(X) \in \mathbb{R}[X] \mid deg(f(X)) \leq n\}$, then $P_n(\mathbb{R})$ has a basis consisting of $n + 1$ polynomials all with degree n.(20%)

II. Let $W_1 = span(\{(0, 2, 2, 3), (0, 4, 4, 6), (1, 2, 3, 6)\})$ and $W_2 = span(\{(-1, 4, 3, 3), (-1, 2, 3, 6), (4, 4, 4, 6)\})$. Find two basis $\mathbb{B}_i \subset W_i$, $i = 1, 2$, such that both $\mathbb{B}_1 \cap \mathbb{B}_2 \subset W_1 \cap W_2$ and $\mathbb{B}_1 \cup \mathbb{B}_2 \subset W_1 + W_2$ are also basis. Is $\mathbb{B}_1 \cup \mathbb{B}_2 \subset \mathbb{R}^4$ a basis of $\mathbb{R}^4$? Why?(20%)

III. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation with real positive eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\sum_{i=1}^{3} \lambda_i = \prod_{i=1}^{3} \lambda_i = 1$. Show that T is diagonalizable and $\mathbb{R}^3$ is T-cyclic.(15%)

IV. Let A and B be the following 2 3x3 matrices

$$A = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

(i) Find the charateristic and the minimal polynomials of A and B over $\mathbb{R}$ and over $\mathbb{C}$.

(ii) Check whether A and B are diagonalizable over $\mathbb{R}$ and over $\mathbb{C}$, and whether $\mathbb{R}^3$ and $\mathbb{C}^3$ are L_A-cyclic and L_B-cyclic.(15%)

V. Let $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R} \setminus \{0, 1\}$, and $A = (a_{ij})_{i,j=1,...,n}$, where $a_{ij} = 1$, if $i \neq j$, and $a_{ii} = \frac{\alpha_i}{\alpha_i - 1}$. Show that $det(A) = 0$ if and only if $\sum_{i=1}^{n} \alpha_i = n - 1$.(15%)

VI. Let $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R}), f \mapsto T(f) = 2f - f'$. Is T diagonalizable? Find the Jordan canonical form of T.(15%)

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I. Prove or disprove the following statements (30%)

1. The series

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\beta} \cos n^{p}}{n^{\alpha}}$$

converges for every $\alpha > 1$, $\beta \geq 1$ and $p \geq 1$. (10%)

2. A sequence of functions $h_n(x)$ is said to converge uniformly to $h(x)$ in a set S, if for given $\varepsilon > 0$ there exists a constant N depending only on ε such that $|h_n(x) - h(x)| < \varepsilon$ for all $x \in S$, for all $n \geq N$.

For the sequence of functions $f_n(x) = (x^2 - 1)^n$, $n \in \mathbb{N}$

define $g_n(x) = \frac{1}{2^n n!} f_n^{(n)}(x)$, $g_0(x) = 1$, then

2a. $f_n(x)$ converges uniformly in $[-\sqrt{2}, 0]$. (10%)

2b. $g_n(x)$ converges uniformly in $[-\sqrt{2}, 0]$. (5%)

2c. $g_n(x)$ satisfies the differential equation

$$((1 - x^2) y')' + n(n+1) y = 0. \text{ (5\%)}$$

II. Evaluate the followings (20%)

3. $z = \min\left\{\sin(x+y)\pi : x^2 + y^2 = 1\right\}$ (10%)

4. $\int_0^{\frac{\pi}{2}} \cos^3 x \ln \sin x \, dx$. (10%)

III. Prove the result (10%)

5. If f is continuous differentiable in $\mathbb{R}$ and the integral $\int_1^{\infty} f(t)/t\,dt$ exists, then for positive a and b we have

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \frac{b}{a}.$$

IV. Solve the differential equation (10%)

6. $u''(t) = 0, u(0) = 0 = u'(0)$.