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(18%) 五. A person 5 ft tall walks at the rate of 3 ft/sec away from a streetlight that is 20 ft above the ground. (a) At what rate is the tip of this person's shadow moving? (b) At what rate is the length of this person's shadow changing?

(12%) 六. Test the following series for convergence or divergence. In case of convergence, determine whether the series converges absolutely.

(a) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$, (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1+n)}{n^2}$, (c) $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$.

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說明：

一. 總共四大題，每大題含十小題（分別依編號[1],[2],[3],...,[10]標示於題句中），每小題計3分（亦即每大題為30分），共120分。

二. 作答時，大題中之小題不可顛倒順序。凡答案錯誤，該小題即不予計分。得分如超過100分，則仍以100分計。

三. 在答案卷上，請清楚標明題號及簡潔的答案（演算或證明步驟不必列出），如下例：

1. [1] *True*
   [2] *False*
   [3] $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$
   ⋮
   [10] $T(M)=\{M \mid M=\begin{bmatrix}a&b\\c&d\end{bmatrix}, a,b,c,d\in\mathbb{R}\}$

2. [1] $L=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$
   [2] $\beta=100$
   [3] *Eigenvalues are* 1, 2, 3, 4.
   [4] $x=\begin{bmatrix}1\\2\\3\end{bmatrix}$
   ⋮
   [10] $A^{-1}=\begin{bmatrix}4&3\\2&1\end{bmatrix}$

1. (a) Let $T: \mathrm{V}\to\mathrm{W}$ be a linear transformation, where V, W are vector spaces with bases $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ respectively. Suppose $T(\mathbf{v}_1)=\mathbf{w}_1+\mathbf{w}_2+\mathbf{w}_3$, $T(\mathbf{v}_2)=\mathbf{w}_2+\mathbf{w}_3$, and $T(\mathbf{v}_3)=\mathbf{w}_3$. Find the matrix $\mathbf{A}$ [1] for T using these basis vectors. What input vector $\mathbf{v}$ [2] gives $T(\mathbf{v})=\mathbf{w}_1$? What are $T^{-1}(\mathbf{w}_1)$ [3], $T^{-1}(\mathbf{w}_2)$ [4], and $T^{-1}(\mathbf{w}_3)$ [5]? Find all $\mathbf{v}$'s [6] that satisfy $T(\mathbf{v})=\mathbf{0}$. (b) Suppose $T:\mathrm{V}\to\mathrm{V}$ with $T(\mathbf{M})=\begin{bmatrix}1&0\\0&0\end{bmatrix}\mathbf{M}\begin{bmatrix}0&0\\0&1\end{bmatrix}$, where V contains all 2-by-2 matrices $\mathbf{M}$. Find a matrix $\mathbf{M}$ [7] with $T(\mathbf{M})\neq\mathbf{O}$ (the zero matrix). Describe all matrices $\mathbf{M}$ [8] with $T(\mathbf{M})=\mathbf{O}$ (the kernel of T) and all output matrices $T(\mathbf{M})$ [9] (the range of T). Suppose $S:\mathrm{V}\to\mathrm{V}$ with $S(\mathbf{M})=\mathbf{AMB}$, where $\mathbf{A}$ and $\mathbf{B}$ are invertible 2-by-2 matrices. Then $S^{-1}(\mathbf{M})=$ [10].

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2. Let $\mathbf{L}$ be a *unit* lower triangular matrix (the diagonal elements of $\mathbf{L}$ are all ones) and $\mathbf{U}$ an upper triangular matrix. (a) Assume $\mathbf{A} = \mathbf{LU}$ and $\mathbf{B} = \mathbf{UL}$. Determine the following statements as being true or false: (i) $\mathbf{A}$ and $\mathbf{U}$ have the same column space [1]. (ii) $\mathbf{A}$ and $\mathbf{U}$ have the same nullspace [2]. (iii) $\det(\mathbf{A}) = \det(\mathbf{U})$ [3]. Express $\mathbf{B}$ [4] in terms of $\mathbf{A}$ and $\mathbf{L}$ by eliminating $\mathbf{U}$. If $\mathbf{A}$ is singular, so is $\mathbf{B}$, true or false [5]? What is the relation [6] between the eigenvalues of $\mathbf{A}$ and those of $\mathbf{B}$? (b) Find the solution of $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 11 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ 10 \\ 14 \end{bmatrix}$, by the following procedure: (i) First factor $\mathbf{A}$ into the product of $\mathbf{L}$ [7] and $\mathbf{U}$ [8]. (ii) Then solve $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$ [9] by forward substitution. (iii) Finally solve $\mathbf{Ux} = \mathbf{y}$ to get $\mathbf{x}$ [10] by back substitution.

3. Consider the real n-by-n matrix $\mathbf{A} = \mathbf{I} + 2\mathbf{u}\mathbf{u}^{\mathsf{T}}$, where $\mathbf{u} \in \mathbb{R}^n$ (a column vector with n real components) and $\|\mathbf{u}\|_2 = \sqrt{\mathbf{u}^{\mathsf{T}}\mathbf{u}} = 1$ ($\mathbf{u}^{\mathsf{T}}$, a row vector, is the transpose of $\mathbf{u}$). (a) Compute $\mathbf{Au}$ and find an eigenvalue [1] of $\mathbf{A}$. For this fixed $\mathbf{u}$, how many [2] linearly independent vectors (up to a scalar multiple) $\mathbf{x} \in \mathbf{R}^n$ can you find such that $\mathbf{u}^{\mathsf{T}}\mathbf{x} = 0$? Now, by choosing such $\mathbf{x}$ and computing $\mathbf{Ax}$, determine all the n eigenvalues [3] of $\mathbf{A}$. (b) Let $\mathbf{E} = \mathbf{I} - \sigma\mathbf{u}\mathbf{v}^T$, $\mathbf{F} = \mathbf{I} - \tau\mathbf{u}\mathbf{v}^T$, where σ, τ are real and *nonzero*, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$. Then $\mathbf{EF} = \mathbf{I}$ if $\mathbf{v}^T\mathbf{u} =$ [4]; if $\sigma\mathbf{v}^T\mathbf{u} =$ [5], then $\mathbf{E}$ is singular. Can $\mathbf{A}$ be singular for some $\mathbf{u}$ [6]? Why? Find $\mathbf{A}^{-1}$ [7] if $\mathbf{A}$ is invertible. What can you say about the Jordan form of $\mathbf{A}$ [8]? (c) Let $\mathbf{G} = \mathbf{I} + \mathbf{u}\mathbf{v}^{\mathsf{T}}$, where $\mathbf{u}, \mathbf{v}$ are defined as in part (b). Give the Jordan form of $\mathbf{G}$ (i.e., describe $\mathbf{J}$ such that $\mathbf{G} = \mathbf{XJX}^{-1}$ for some nonsingular $\mathbf{X}$) in two different cases: (i) $\mathbf{v}^{\mathsf{T}}\mathbf{u} \neq 0$ [9] (ii) $\mathbf{v}^{\mathsf{T}}\mathbf{u} = 0$ [10].

4. (a) Matrices $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ are similar, true or false [1]? (b) Which of the following matrices is/are not diagonalizable [2]? (i) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} i & i \\ i & i \end{bmatrix}$, $i = \sqrt{-1}$ (vii) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. (c) Let $\mathbf{A} = \begin{bmatrix} 2 & \beta \\ 1 & 0 \end{bmatrix}$. If $\beta \neq$ [3], then $\mathbf{A}$ is diagonalizable; if $\beta =$ [4], then $\mathbf{A}$ is orthogonally diagonalizable (i.e., $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{\mathsf{T}}$, where $\mathbf{Q}$ is orthogonal and Λ is diagonal). (d) Let $\mathbf{P}$ ($\neq \mathbf{O}$) be an orthogonal projection matrix (i.e., $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}^{\mathsf{T}} = \mathbf{P}$) from $\mathbb{R}^n$ onto a subspace of dimension $k < n$. What is the minimal polynomial [5] for $\mathbf{P}$? What are the n eigenvalues [6] of $\mathbf{P}$? (e) Find the four eigenvalues of: (i) $\mathbf{A} = \begin{bmatrix} 7 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 7 \end{bmatrix}$ [7] (Hint: Consider the rank of $\mathbf{A} - \lambda\mathbf{I}$ for some λ.) (ii) $\mathbf{B} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$ [8] (Hint: $\mathbf{B}$ is skew-symmetric and orthogonal.) (f) Determine the 2-by-2 matrix $\mathbf{A}$ [9] such that $f(x_1, x_2) =$

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第一大題為選擇題 計14小題，對每一小題 請選一個最適當(或最接近)的選項，不需寫計算過程或証明，每小題3分，答錯不倒扣；第二至第六大題，必須詳細敘述每一步的計算過程或使用的定理。

(42%) 1. The distance from the point $(-1, 4)$ to the circle $x^2+y^2-4x=0$ is

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

2. Let y be a differentiable function of x such that $2xy+x^2y^2=3$. Which of the following functions, where defined, is equal to $\frac{dy}{dx}$?

(a) $-xy$ (b) $-\frac{xy^2}{1+x^2y}$ (c) $-\frac{x^2y}{1+xy^2}$ (d) $-\frac{x}{y}$ (e) $-\frac{y}{x}$

3. Suppose $\lim_{x\to 1} f(x)=3$. Then for any positive number ε there is a positive number δ such that, whenever $0<|x-1|<\delta$,

(a) $|x-3|<\varepsilon$ (b) $|f(x)-1|<\varepsilon$ (c) $|f(1)-3|<\varepsilon$ (d) $|f(x)-3|<\varepsilon$ (e) $|f(x)-\varepsilon|<3$

4. If $g\left(\frac{1+3x}{2}\right)=x-1$, $-\infty<x<\infty$, then $g\left(\frac{2+5y}{3}\right)$ must equal:

(a) $y-1$ (b) $\frac{5y-1}{3}$ (c) $\frac{3+5y}{2}$ (d) $\frac{1+10y}{9}$ (e) $\frac{10y-8}{9}$

5. If, for all x, $f'(x)=(x-5)^4(x-3)^3$, it follows that the function f has

(a) a relative minimum at $x=3$ (b) a relative maximum at $x=3$

(c) both a relative minimum at $x=3$ and a relative maximum at $x=5$

(d) neither a relative maximum nor a relative minimum

(e) relative minima at $x=3$ and at $x=5$.

6. If $x_n=\begin{cases}\frac{n+3}{3n} & \text{if } n \text{ is odd}\\ \frac{2n+3}{6n} & \text{if } n \text{ is even}\end{cases}$, then $\lim_{n\to\infty} x_n=$

(a) $1/3$ (b) 1 (c) 2 (d) 3 (e) does not exist

7. Let $z=f(u,v)$, where f has continuous first partial derivatives. If $u=2y-x$ and $v=y-x$, then $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=$

(a) $-\frac{\partial z}{\partial v}$ (b) $\frac{\partial z}{\partial v}$ (c) $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$ (d) $\frac{\partial z}{\partial u}$ (e) 0

8. Which of the following functions does not satisfy the hypotheses of Rolle's theorem on the interval $[0,1]$?

(a) $g(x)=\frac{x^2-x}{2x-1}$ (b) $g(x)=x^2-x$ (c) $g(x)=x^3-x^2$ (d) $g(x)=(x-1)(e^x-1)$ (e) $g(x)=\frac{x^2-x}{x+1}$

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一、9. What is the number of points of discontinuity of the function

$$g(x) = \begin{cases} x + 1/2 & \text{if } x \le 0 \\ \frac{1}{2} & \text{if } 0 < x \le 2 \\ 3 - x & \text{if } 2 < x \le 3 \\ (3-x)^2 & \text{if } 3 < x \end{cases}$$

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

10. If f and g are differentiable functions such that $f(x) = e^{2x} \cdot g(x)$, $g(0) = 2$, and $g'(0) = -1$, then $f'(0) =$

(a) -1 (b) 0 (c) 1 (d) 2 (e) 3

11. If $f(x) = \frac{d}{dx} g(x)$ is integrable on the closed interval $[0,1]$, then $\int_0^1 g(x) f(x)\,dx =$

(a) $\frac{f^2(1) - f^2(0)}{2}$ (b) $\frac{g(1)f(1) - g(0)f(0)}{2}$ (c) $g(1)f(1) - g(0)f(0)$ (d) $g(1) - g(0)$ (e) $\frac{g^2(1) - g^2(0)}{2}$

12. If $\lim_{x \to 0} \frac{f(\sin x + 3x) - f(0)}{x} = \frac{1}{5}$, then $f'(0) =$

(a) $\frac{1}{20}$ (b) $\frac{1}{5}$ (c) $\frac{1}{4}$ (d) $\frac{4}{5}$ (e) 5

13. Let $f(x) = x^3 + 3x$, $0 \le x < \infty$, and let g be the inverse function of f. Then $g'(4) =$

(a) $\frac{1}{72}$ (b) $\frac{1}{30}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$ (e) 1

14. If f is differentiable on $(-\frac{1}{2}, \frac{1}{2})$, then $\lim_{h \to 0} \frac{1}{h} \int_0^{h/2} f(x)\,dx$ must equal:

(a) $f'(0)/2$ (b) $f'(0)$ (c) $f(0)/2$ (d) $f(0)$ (e) $\frac{f(0) + f'(0)}{2}$

(20%) 二、Each of the following integrals represents the area of a region in a Cartesian coordinate plane. Sketch the region. Express the area of the region as a double integral with the order of integration reversed. Then evaluate these new integrals.

(1) $\int_0^4 \int_{-\sqrt{4-y}}^{(4-y)/2} dx\,dy$ (2) $\int_{-1}^1 \int_{u^2}^1 dv\,du$

(3) $\int_0^{3/2} \int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt$ (4) $\int_0^2 \int_0^{4-w^2} 2w\,dz\,dw$

(10%) 三、Use Taylor's formula to find a quadratic polynomial that approximates

$$f(x,y) = \sin x \sin y$$

near the origin. How accurate is the approximation if $|x| \le 0.1$ and $|y| \le 0.1$?

(8%) 四、Prove or disprove that, if $t > 0$, then $t > \ln(1+t)$.

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