考試科目	線性代數	所別	應用數學系	考試時間	4 月 20 日 星期 六 上 下 午第 2 節

4. Let $V = M_{3\times3}(R)$ be the vector space of all 3×3 real matrices. Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}.$$

Define a linear operator T on V by $T(B) = AB$ for $B \in V$.

(1) Find the nullspace (kernel) of T.

(2) Evaluate $2A^5 - 11A^4 - 13A^3 + 100A^2 - 50A + +30I$.

(3) Show that the minimal polynomial of T is the same as the minimal polynomial of A.

(4) Is T diagonalizable? Why? (20%)

備考	試題隨卷繳交

考試科目	微積分	所別	數學	考試時間	4月20日上午第一節 星期六

1. (*a*) Show that for every real x the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges.(5%)
   (*b*) Denoting $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$, then $f(x)$ is continuous in $[0,\pi]$.(5%)
   (*c*) Prove that $\int_0^{\pi} f(x)dx = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$.(5%)

2. Evaluate the limits
   (*a*) $\lim_{x\to 0^+} \left(\ln \frac{1}{x}\right)^x$.(7%)
   (*b*) $\lim_{h\to 0} \frac{1}{h} \int_h^{2h} \left(\frac{\sin^{-1} x}{x}\right)^{\frac{1}{x^2}} dx$.(10%)

3. Let $F(x) = \int_0^x f(t)dt$. Determine a formula (or formulas) for computing $F(x)$ for all real x, if f is defined as follows
   (*a*) $f(t) = \frac{2t+5}{t^2+2t-3}$. (7%) (*b*) $f(t) = \frac{1}{\cos t + \sin t}$. (8%)

4. Let $f(x,y,z) = x^2 + y^2 + z^2$.
   (*a*) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$. (5%)
   (*b*) What is the maximum rate of increase of f at $(1,-1,2)$. (5%)
   (*c*) Find the minimum of the values $\frac{|x-y+2z|}{\sqrt{6}}$ on the sphere $f(x,y,z) = 6$. (8%)

5. Calculate the iterated integral $\int_0^1 \int_0^y (x\sin x + y^2 \cos x)dxdy$. (10%)

6. Prove or disprove the following statements.
   (*a*) If f_n and f are C^1 – functions and f_n converges uniformly to f in S, then f_n' converges to f' in S. (10%)
   (*b*) Let f and g be two functions continuous in a closed interval $[a,b]$ and having derivatives in open interval (a,b). Then there exists c in (a,b) such that

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a)). \quad (15\%)$$

考試科目	線性代數	所別	應用數學系	考試時間	4月20日上下午第2節 星期六

1. Prove or disprove: (48%)

   (1) If a square matrix A has orthonormal columns, then A has orthonormal rows.

   (2) Let A be a $m \times n$ matrix. Then $\operatorname{rank} A^T A = \operatorname{rank} A$.

   (3) The matrix $A_{7\times7} = B_{7\times5} C_{5\times7}$ has no inverse.

   (4) There exists a linear transformation on R^2 which maps a rectangle to an ellipse.

   (5) Every diagonal entry of a positive definite matrix must be positive.

   (6) There exist no square matrices A and B such that $AB - BA = I$.

   (7) For every real symmetric $n \times n$ matrix A, there is a real constant k such that the matrix $A + kI_n$ is positive definite.

   (8) For every $n \times n$ matrix A, there is a constant k such that $A + kI_n$ is nonsingular.

2. Let $A_{m\times n} x_{n\times1} = b_{m\times1}$ be a consistent linear system with real coefficients.

   (1) Show that this system has one and only one solution x_0 in $RS(A)$, the row space of A. (Hint: What is the orthogonal complement of $RS(A)$?) (6%)

   (2) If x_0 is the solution in $RS(A)$ and x_1 is any other solution of $Ax = b$, show that $\|x_0\| \le \|x_1\|$. The vector x_0 is called the minimal solution of the linear system $Ax = b$. (6%)

   (3) Find the minimal solution of $\begin{cases} x + 2y + z = 4 \\ x - y + 2z = -11. \\ x + 5y = 19 \end{cases}$

   (Hint: Find one solution, then project it into $RS(A)$). (8%)

3. (1) On the surface $-x_1^2 + x_2^2 - x_3^2 + 10x_1x_3 = 1$, find the two points closest to the origin. (6%)

   (2) Find the maximum and minimum of $x_1^2 + 2x_2^2 + 3x_3^2 + x_1x_2 + 2x_1x_3 + 3x_2x_3$ on the unit sphere. (6%)

備考	試題隨卷繳交