考試科目 Course	線性代數	系級	應數系	日期 Date, Period	6 月 10 日 第 1 節	試題編號 Course No.

1. In R^2, let $i_0 = (\cos\theta)i + (\sin\theta)j$, $j_0 = -(\sin\theta)i + (\cos\theta)j$.
(20%) Let T_0 be the operator that carries i_0 into j_0 and j_0 into 0.
(1) Calculate the matrix representation of T_0 relative to the basis $\{i, j\}$ for R^2.
(2) Show that $T_0^2 = 0$.

2. Find the minimal solution of the system
(20%)
$$\begin{cases} x + 2y + z = 4 \\ x - y + 2z = -11 \\ x + 5y \quad = 19 \end{cases}$$

3. Consider the operator $T : R^2 \to R^2$ given by
(20%)
$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b \end{bmatrix}.$$
Show that $U = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is T-invariant but that U has no T-invariant complement in R^2.

4. Suppose that $A \in M_{n\times n}(F)$ can be written in the form
(20%)
$$A = \begin{bmatrix} B & X \\ O & C \end{bmatrix},$$
where B and C are square matrices, respectively.
Prove that $\det(A) = \det(B)\cdot\det(C)$.

5. Let A be a nonsingular symmetric $n\times n$ matrix with
(20%) eigenvalues λ_i $(i = 1, 2, \cdots, n)$. (Recall that the trace of $A = [a_{ij}]_n$, written $tr(A)$, equals $\sum_{i=1}^{n} a_{ii}$.) Prove that
$$tr(A^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1}.$$

考試科目 Course	分析概論	系級	應數	日期 Date, Period	6 月 10 日 第 2 節	試題編號 Course No.

1. (10%) Let $f : [0,1] \to R$ be continuous and differentiable on (a,b). Assume that $f(a) = 0, f(b) = -1$, and $\int_a^b f(x)dx = 0$. Prove that $f'(c) = 0$ for some $c \in (a,b)$.

2. (10%)Let $f : [0,1] \to R$ be defined by

$$f(x) = \begin{cases} 0 & if \quad x \text{ is irrational} \\ \frac{1}{q} & if \quad x = \frac{p}{q} \end{cases}$$

where $p,q \geq 0$ with no common factor.
Is f integrable on $[0,1]$? .

3. (10%)If f is a real, three times differentiable function on $[-1,1]$ such that

$$f(-1) = 0, f(0) = 0, f(1) = 1, f'(0) = 0$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$ by Taylor?s theorem.

4. (10%)By using some transformation, find

$$\int_A xy\sin(x^2 - y^2)dxdy$$

where

$$A = \{(x,y)|0 < y < 1, x > y, x^2 - y^2 < 1\}$$

5. (10%)If $f \in L^2(R), g \in L^3(R)$, and $g \in L^6(R)$, show that $fgh \in L^1(R)$ and $\|fgh\|_1 \leq \|f\|_2 \|g\|_3 \|h\|_6$(10%)

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6. (10%)Let $C([0,1])$ be the set of real continuous functions on $[0,1]$. Show that the complement of the following set

$$A = \left\{ f \in C([0,1]) \mid 0 < \int_0^1 f(x)dx < 3 \right\}$$

is closed.

7. (10%)Prove that

$$\lim_{n\to\infty} \int_0^1 e^{-nx^2} x^p dx = 0, \text{ if } p > -1$$

by Lebesque convergence theorem.

8. (10%)Let (M,d) be a metric space. If A is compact in M and B is closed in M, and $A \cap B = \phi$. Show that there is an $\delta > 0$ such that $d(x,y) > \delta$ for all $x \in A$ and $y \in B$.

9. (10%)Let $k(x,y)$ be a continuous real-valued function on the square $S = [0,1] \times [0,1]$. Assume that $|k(x,y)| < 1$ for each $(x,y) \in S$. Let $A : [0,1] \to R$ be continuous. Prove that there is a unique continuous real-valued function $f(x)$ on $[0,1]$ such that

$$f(x) = A(x) + \int_0^1 k(x,y)f(y)dy$$

by contraction mapping theorem.

10. (10%)Let $u_n > 0$, $n \in N$, show that

$$\limsup \sqrt[n]{u_n} \le \limsup \frac{u_{n+1}}{u_n}$$