科技部補助專題研究計畫成果報告
期末報告

SIS模型之旅行波解

計 畫 類 別 ： 個別型計畫
計 畫 編 號 ： MOST 104-2115-M-004-002-
執 行 期 間 ： 104年08月01日至105年07月31日
執 行 單 位 ： 國立政治大學應用數學學系

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中 華 民 國 105 年 10 月 18 日

中 文 摘 要 ： 本計畫中，我們是探討流行病感染者痊癒後不具免疫力的一個具擴散性之SIS模型，此模型沒有比較原理，我們證明了存在連結流行均衡點及非流行均衡點之行進波解。

中文關鍵詞： SIS模型, 行進波, 傳遞速度, 上/下解, Schauder定點定理

英 文 摘 要 ： We study a diusive SIS model for a disease that the infectives recover with no
immunity against reinfection. Such a SIS model does not enjoy the comparison principle. We analytically show that there exists a family of traveling waves connecting the endemic equilibrium with the disease-free equilibrium.

英文關鍵詞： SIS model, Traveling wave, Spreading speed, Super/sub-solution, Schauder fixed point theorem

Epidemic waves for the diffusive SIS model*

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October 17, 2016

Abstract

We study a diffusive SIS model for a disease that the infectives recover with no immunity against reinfection. Such a SIS model does not enjoy the comparison principle. We analytically show that there exists a family of traveling waves connecting the endemic equilibrium with the disease-free equilibrium.

Key Words: SIS model. Traveling wave. Spreading speed. Super/sub-solution, Schauder fixed point theorem

1 Introduction

In this paper, we consider a diffusive SIS model for a disease that the infectives recover without immunity against reinfection. To be precise, let $S = S(x,t)$ represent the number at time t and position x of individuals who are susceptible to the disease, and $I = I(x,t)$ denote the number at time t and position x of infected individuals who can spread the disease by contacting with susceptible individuals. Then the model reads:

$$S_t = \delta S_{xx} + \mu\Lambda - \beta SI - \mu S + \gamma I, \tag{1.1a}$$

$$I_t = I_{xx} + \beta SI - \mu I - \gamma I - \kappa I. \tag{1.1b}$$

Here the parameters Λ, μ, β, and γ are positive constants. Moreover, the constant $\mu\Lambda$ is the recruitment rate of the susceptible population S, β is the contact rate, γ is the recovery rate of the infective population, μ is the natural death rate for both the susceptible and the infective population, and κ is the rate of the infective population dying from infection. The constant Λ can be interpreted as a carrying capacity, or maximum possible population size. Finally, the parameter δ is the ratio of the diffusion rate of the susceptible population to that of the infective population. This model can be used to describe transmission of diseases such as sexual transmitted disease, plague, and meningitis.

*This work is partially supported by National Science Council of the Republic of China.

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Note that if the quantity

$$R_0 := \frac{\beta\Lambda}{\gamma + \mu + \kappa} < 1$$

then system (1.1) has only one equilibrium point: $(\Lambda, 0)$, which is called the disease-free equilibrium. On the other hand, if $R_0 > 1$ then system (1.1) has the second equilibrium point: (s^*, i^*), where

$$s^* := \frac{\mu + \gamma + \kappa}{\beta} \quad \text{and} \quad i^* := \frac{\mu(\Lambda - s^*)}{\mu + \kappa},$$

which is called the endemic-equilibrium. Furthermore, with a local analysis, one can verify that the disease-free equilibrium $(\Lambda, 0)$ is a saddle point of the kinetic equation of system (1.1)(i.e., system (1.1) without diffusion), while the endemic-equilibrium (s^*, i^*) is a stable node of the kinetic equation of system (1.1). This observation suggests that if $R_0 < 1$, the infection should die out, while if $R_0 > 1$, the infection will spread. In epidemiology, the quantity R_0 is called the basic reproduction number. Since we are concerned with spread of the infection, throughout this paper, we always assume that $R_0 > 1$.

Numerical simulations show that by locally introducing an amount of the infective population into the area which is inhabited by the susceptible population at the level of the carrying capacity Λ, the corresponding solution evolves into a pair of diverging travelling waves propagating outwards from the initial zone. In the present paper, we shall analytically show the existence of traveling wave solutions of system (1.1).

A traveling wave solution of system (1.1) is a solution of system (1.1) of the form

$$(S(x,t), I(x,t)) = (s(z), i(z)), \quad z = x + ct,$$

with the boundary condition $(s,i)(+\infty) = (s^*, i^*)$ and $(s,i)(-\infty) = (\Lambda, 0)$. Here the wave speed c is a constant to be determined and the wave profile $(s,i) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ is a pair of nonnegative functions. Upon substituting the ansatz on (s,i) into (1.1), the governing system for (s,i) reads:

$$\delta s'' - cs' + \mu(\Lambda - s) - \beta si + \gamma i = 0, \tag{1.2a}$$
$$i'' - ci' + \beta si - (\mu + \gamma + \kappa)i = 0 \tag{1.2b}$$

on $\mathbb{R}$, together with the boundary conditions

$$(s,i)(+\infty) = (s^*, i^*), \quad (s,i)(-\infty) = (\Lambda, 0). \tag{1.3}$$

Here the prime indicates differentiation with respect to z. Now we are in a position to state the main result on the existence of traveling waves of system (1.1) as follows:

Theorem 1.1 (Existence of traveling waves)

(I) *For each* $c < c_{min} := 2\sqrt{\beta\Lambda - \gamma - \mu - \kappa}$, *there are no nonnegative solutions* (s,i) *of system* (1.2)-(1.3).

(II) *For each* $c > c_{min}$, *system* (1.2)-(1.3) *admits a nonnegative solution* (s,i) *with the following properties:*

(i) $\gamma/\beta < s < \Lambda$ *and* $i > 0$ *over* $\mathbb{R}$.

(ii) *There exists a* $\gamma^* > 0$ *such that there hold*

(a) *if* $\gamma \in (0, \gamma^*)$, *then the solution* (s, i) *approaches* (s^*, i^*) *monotonically for large* z.

(b) *if* $\gamma > \gamma^*$, *then the solution* (s, i) *has exponentially damped oscillations about* (s^*, i^*) *for large* z.

(iii) *We have* $i(z) = \mathcal{O}(e^{\lambda z})$ *as* $z \to -\infty$, *where* λ *is given by*

$$\lambda = \lambda(c) := \frac{1}{2} \cdot \left[c - \sqrt{c^2 - 4(\beta\Lambda - \gamma - \mu - \kappa)} \right]. \tag{1.4}$$

We make two comments on Theorem 1.1. First, the minimal speed c_{min} of traveling waves of system (1.1) is independent of the ratio δ of the diffusion rates. Second, due to the lack of uniform bound of the i-component of traveling wave for c close to c_{min}, we are unable to show the existence of critical waves (i.e., waves with speed $c = c_{min}$). We left this question for our future study.

Finally, we outline the method for the proof of main results. We will follow the framework of our previous work [7] to establish Theorem 1.1 whose idea is based on [3]. Note that our previous work [7] can only be applied to system (1.1) with $\gamma = 0$. There are two main steps for the methods in [7]. First, we need to construct a pair of *coupled* super/sub-solutions of system (1.1), then use this set of super/sub-solutions to derive the existence of the solution of the truncated problem associated with system (1.2), and then, by passing to the limit, get a candidate solution (s, i) for the traveling wave solution of system (1.2). Second, in order to verify that the candidate solution (s, i) satisfies the boundary condition at the infinity, we need to derive the estimates of the derivative of (s, i) and the boundedness of the i-component, and then apply the LaSalle's theorem to get that (s, i) satisfies the boundary condition at infinity. The first step is different from that in [7] since we use the super-solution to set up the boundary condition for the truncated problem. The second step is a slight modification of that in [7]. Hence for the proof of the second step, we will only sketch the main ingredients, and refer the readers to [7] for more details. We remark that due to the lack of comparison principle of system (1.1), the construction of the sub-solution is based on the super-solution, not on the traveling waves.

This paper is organized as follows. In Sec. 2, we first construct the coupled pairs of super/sub-solutions, and then use this set of super/sub-solutions and Schauder fix point theorem to establish the solution of truncated problem of system (1.1) on the finite interval $[-l, l]$. Finally, by passing to the limit $l \to \infty$, we obtain a solution (s, i) of system (1.1) on $\mathbb{R}$ with the condition $(s, i)(-\infty) = (\Lambda, 0)$ which is a candidate solution for traveling waves of system (1.1). In Sec. 3, we verify that the candidate solution (s, i) obtained in Sec. 2 is indeed a traveling wave solutions of system (1.1). Finally, some auxiliary lemmas are given in the appendix.

2 Property of waves and construction of a candidate of traveling waves

2.1 The minimal speed and decay rate of waves

We first establish the assertion of Theorem 1.1 (I) and the decay rate of the i-component of waves near infinity.

Lemma 2.1 *Suppose that (s, i) is a nonnegative solution of system (1.2)-(1.3). Then we have*

(i) $c \geq c_{min}$, *and*

(ii) *For $c > c_{min}$, $i(z) = \mathcal{O}(e^{\lambda z})$ as $z \to -\infty$ where λ is given by*

$$\lambda = \frac{1}{2} \cdot \left(c \pm \sqrt{c^2 - 4(\beta\Lambda - \gamma - \mu - \kappa)} \right).$$

Proof. Linearizing (1.2) around $(\Lambda, 0)$ yields the equations

$$\delta s'' - cs' - \mu s - (\beta\Lambda - \gamma)i = 0, \tag{2.1a}$$

$$i'' - ci' + (\beta\Lambda - \mu - \gamma - \kappa)i = 0. \tag{2.1b}$$

Note that (2.1b) has two eigenvalues

$$\lambda_1 = \frac{1}{2} \cdot \left(c - \sqrt{c^2 - 4(\beta\Lambda - \gamma - \mu - \kappa)} \right), \quad \lambda_2 = \frac{1}{2} \cdot \left(c + \sqrt{c^2 - 4(\beta\Lambda - \gamma - \mu - \kappa)} \right).$$

For contradiction, we assume $|c| < 2\sqrt{\beta\Lambda - \gamma - \mu - \kappa}$ holds. Then λ_1 and λ_2 form a complex conjugate pair. This suggests that $i(z)$ cannot be of the same sign for z for large $-z$, a contradiction. Hence we have $|c| \geq 2\sqrt{\beta\Lambda - \gamma - \mu - \kappa}$. Next we suppose that $c \leq -2\sqrt{\beta\Lambda - \gamma - \mu - \kappa}$. Then we have $\lambda_i > 0$, $i = 1, 2$, and so $i(z)$ is unbounded as $z \to -\infty$, which is a contradiction. Taken together, we can conclude $c \geq c_{min} = 2\sqrt{\beta\Lambda - \gamma - \mu - \kappa}$, which completes the proof of assertion (i).

Finally, the assertion (ii) follows from the above linearized equation and the definitions of λ_1 and λ_2. This completes the proof of this lemma. □

In the remaining of this section, we will construct a candidate of non-critical waves and hence we always assume that $c > c_{min}$.

2.2 Super/sub-solutions

In this subsection, we will construct a pair of super- and sub-solutions $(s^\pm, i^\pm)$. To begin with, we give the definition of super- and sub-solutions of (1.2).

Definition 2.1 *(s^+, i^+) and (s^-, i^-) are called a pair of super- and sub-solutions of (1.2) if s^+, i^+, s^-, i^- are nonnegative continuous functions and satisfy*

$$\begin{aligned}
&\delta(s^+)''(z) - c(s^+)'(z) + \mu(\Lambda - s^+(z)) - \beta s^+(z)i^-(z) \leq 0, \\
&\delta(s^-)''(z) - c(s^-)'(z) + \mu(\Lambda - s^-(z)) - \beta s^-(z)i^+(z) \geq 0, \\
&\big(i^+\big)''(z) - c\big(i^+\big)'(z) + \beta s^+(z)i^+(z) - (\gamma + \mu + \kappa)i^+(z) \leq 0, \\
&\big(i^-\big)''(z) - c\big(i^-\big)'(z) + \beta s^-(z)i^-(z) - (\gamma + \mu + \kappa)i^-(z) \geq 0
\end{aligned}$$

except for finitely many points of z in $\mathbb{R}$.

The idea of the construction of the super/sub-solutions is motivated by [3]. Specifically, we first construct the s-component of the super-solution s^+. Then we use s^+ to construct the i-component of the super-solution i^+, which is immediately employed to construct the s-component of the sub-solution s^-. The s^- is in turn used to generate the i-component of the sub-solution i^-.

To construct the super/sub-solutions, we select $0 < \alpha < \min\{c/\delta, \lambda_1\}$ and $0 < \eta < \min\{\alpha, \lambda_2 - \lambda_1\}$ such that

$$c - \delta\alpha > 0, \tag{2.2}$$

$\lambda_1 - \alpha > 0$, $\eta - \alpha < 0$, and $P(\lambda_1 + \eta) < 0$. In view of the fact that $e^{(\lambda_1-\alpha)z} \to 0$ as $z \to -\infty$, there exists $z_0 < 0$ such that

$$e^{(\lambda_1-\alpha)z} \leq \frac{\mu}{\beta}, \forall z \leq z_0.$$

Hence we have

$$\mu e^{\alpha z} \geq \beta i^+(z), \forall z \leq z_0 \tag{2.3}$$

and

$$M := \Lambda e^{-\alpha z_0} > \Lambda. \tag{2.4}$$

Finally, we pick

$$L > \max\left\{\frac{M}{\Lambda}, -\frac{\beta M}{P(\lambda_1+\eta)}\right\}, \tag{2.5}$$

and set $z_1 = -\ln L/\eta$. Note that $z_1 < z_0 < 0$ since $z_0 = -\ln M/\alpha$, $L > M$, and $\eta < \alpha$.

Now we define four nonnegative continuous functions s^+, s^-, i^+, and i^- as follows:

$$\begin{aligned}
s^+(z) &:= \Lambda,\\
s^-(z) &:= \begin{cases} \Lambda - Me^{\alpha z}, & z \leq z_0,\\ 0, & z > z_0, \end{cases}\\
i^+(z) &:= e^{\lambda_1 z},\\
i^-(z) &:= \begin{cases} e^{\lambda_1 z} - Le^{(\lambda_1+\eta)z}, & z \leq z_1,\\ 0, & z > z_1. \end{cases}
\end{aligned}$$

It is obvious that $s^+(z)$ satisfies the inequality

$$\delta(s^+)''(z) - c(s^+)'(z) + \mu(\Lambda - s^+(z)) - \beta s^+(z)i^-(z) \leq 0 \tag{2.6}$$

for all $x \in \mathbb{R}$. In the following, we will show that (s^+, i^+) and (s^-, i^-) are a pair of upper and lower solutions of (1.2) .

Lemma 2.2 *The function $i^+(z)$ satisfies the equation*

$$\big(i^+\big)''(z) - c\big(i^+\big)'(z) + \beta s^+(z)i^+(z) - (\gamma + \mu + \kappa)i^+(z) = 0 \tag{2.7}$$

for all $z \in \mathbb{R}$, where the prime denotes the differentiation with respect to z.

Proof. Since $P(\lambda_1) = 0$, it follows that

$$\big(i^+\big)''(z) - c\big(i^+\big)'(z) + \beta s^+(z)i^+(z) - (\gamma + \mu + \kappa)i^+(z) = P(\lambda_1) \cdot i^+(z) = 0, \forall z \in \mathbb{R}.$$

□

In the sequel, we retain the notation z_0.

Lemma 2.3 *The function $s^-(z)\}$ satisfies the inequality*

$$\delta(s^-)''(z) - c(s^-)'(z) + \mu(\Lambda - s^-(z)) - \beta s^-(z)i^+(z) \geq 0 \tag{2.8}$$

for all $z \neq z_0$.

Proof. For $z > z_0$, the inequality (2.8) follows from $s^-(z) \equiv 0$ in (z_0, ∞). For $z < z_0$, $s^-(z) = \Lambda - Me^{\alpha z}$, and hence $\Lambda - s^-(z) = Me^{\alpha z}$. Together with the fact that $s^-(z) \leq \Lambda$, we can use deduce

$$\begin{aligned} & \delta(s^-)''(z) - c(s^-)'(z) + \mu(\Lambda - s^-(z)) - \beta s^-(z)i^+(z) \\ \geq\ & M\alpha(c - \delta\alpha)e^{\alpha z} + \mu Me^{\alpha z} - \beta\Lambda i^+(z) \\ \geq\ & M\alpha(c - \delta\alpha)e^{\alpha z} + M\big(\mu e^{\alpha z} - \beta i^+(z)\big) \quad \text{(by (2.3))} \\ \geq\ & 0. \quad \text{(by (2.2) and (2.4))} \end{aligned}$$

Hence (2.8) holds. □

In the sequel, we retain the notation z_1 and L.

Lemma 2.4 *The function $i^-(z)$ satisfies the inequality*

$$\big(i^-\big)''(z) - c\big(i^-\big)'(z) + \beta s^-(z)i^-(z) - (\gamma + \mu + \kappa)i^-(z) \geq 0 \tag{2.9}$$

for all $z \neq z_1$.

Proof. For $z > z_1$, the inequality (2.9) follows from $i^-(z) \equiv 0$ in (z_1, ∞). For $z < z_1$, $i^-(z) = i^+(z) - Le^{(\lambda_1+\eta)z}$ and $s^-(z) = \Lambda - Me^{\alpha z}$. Then we have

$$(i^-)'(z) = (i^+)'(z) - (\lambda_1 + \eta)Le^{(\lambda_1+\eta)z},$$
$$(i^-)''(z) = (i^+)''(z) - (\lambda_1 + \eta)^2 Le^{(\lambda_1+\eta)z},$$

and

$$\begin{aligned} & s^-(z)i^-(z) \\ =\ & (\Lambda - Me^{\alpha z})\left(i^+(z) - Le^{(\lambda_1+\eta)z}\right) \\ \geq\ & \Lambda i^+(z) - Me^{(\alpha+\lambda)z} - \Lambda Le^{(\lambda_1+\eta)z}. \end{aligned}$$

Together with (2.7) and definition of P, we get

$$\begin{aligned} & (i^-)''(z) - c(i^-)'(z) + \beta s^-(z)i^-(z) - (\gamma + \mu + \kappa)i^-(z) \\ \geq\ & e^{(\lambda_1+\eta)z}[-P(\lambda_1 + \eta)L - \beta Me^{(\alpha-\eta)z}] \\ \geq\ & 0. \quad \text{(by } e^{(\alpha-\eta)z} \leq 1 \text{ and (2.5))} \end{aligned}$$

This completes the proof of this lemma. □

2.3 A truncated problem

In this subsection, we will use the super/sub solutions established in Sec. 2.2 to construct the solutions of the truncated problem of system (1.2)-(1.3). With the aid of the solution of the truncated problem, we can use the limiting process to obtain a solution (s,i) of system (1.2) satisfying $(s,i)(-\infty)=(\Lambda,0)$ which can be a good candidate of traveling wave solutions of system (1.1).

Let $l>z_1$. We consider the following truncated problem

$$\delta s''-cs'+\mu(\Lambda-s)-\beta si+\gamma i=0 \quad \text{in } (-l,l), \tag{2.10a}$$
$$i''-ci'+\beta si-(\gamma+\mu+\kappa)i=0 \quad \text{in } (-l,l), \tag{2.10b}$$

together with the boundary conditions

$$(s,i)(-l)=(s^+,i^+)(-l), \quad (s,i)(l)=(s^+,i^+)(l). \tag{2.11}$$

In the remaining of this subsection, we will employ the Schauder fixed point theorem to establish the existence of solutions of (2.10)-(2.11). To do this, we set $I_l:=[-l,l]$ and $X:=C(I_l)\times C(I_l)$. Define the working space

$$E:=\{(s,i)\in X|\ s^-\le s\le s^+\equiv\Lambda \text{ and } i^-\le i\le i^+ \text{ in } I_l\},$$

which is a closed convex set in the Banach space X equipped with the norm $\|(f_1,f_2)\|_X=\|f_1\|_{C(I_l)}+\|f_2\|_{C(I_l)}$. Since s^- and i^- are nonnegative, it follows that $s\ge 0$ and $i\ge 0$ for any $(s,i)\in E$. Next, we define the mapping $\mathcal{F}E\to E$ as follows: given $(s_0,i_0)\in E$, set

$$\mathcal{F}(s_0,i_0):=(s,i),$$

where (s,i) is the solution of the boundary value problem

$$\delta s''-cs'+\mu(\Lambda-s)-\beta si_0+\gamma i_0=0 \quad \text{in } (-l,l), \tag{2.12a}$$
$$i''-ci'+\beta s_0i_0-(\gamma+\mu+\kappa)i=0 \quad \text{in } (-l,l), \tag{2.12b}$$
$$(s,i)(-l)=(s^+,i^+)(-l), \quad (s,i)(l)=(s^+,i^+)(l). \tag{2.12c}$$

Note that any fixed point of $\mathcal{F}$ is a solution of the problem (2.10)-(2.11). Hence in order to solve the problem (2.10)-(2.11), it suffices to verify that the mapping $\mathcal{F}$ satisfies the condition of the Schauder fixed point theorem. We will do this in the remaining part of this subsection.

Lemma 2.5 *The mapping $\mathcal{F}$ is well-defined; that is, for a given $(s_0,i_0)\in E$, there exists a unique solution (s,i) to the boundary value problem (2.12). Moreover, $s^-\le s\le s^+$ and $i^-\le i\le i^+$ in I_l.*

Proof. Since system (2.12) is not a coupled system and the equations (2.12a) and (2.12b) are inhomogeneous linear equations, the existence and uniqueness to the boundary value problem (2.12) can be easily obtained by [8, Theorem 3.1 of Chapter 12]. Moreover, since $\delta s''-cs'-(\mu+\beta i_0)s=-\mu\Lambda-\gamma i_0\le 0$ on $(-l,l)$ and $s(\pm l)=s^+(\pm l)=\Lambda>0$, it follows from the maximum principle that $s>0$ over I_l. Similarly, one can deduce that $i>0$ over I_l.

Next we claim that $s^- \leq s \leq s^+$ in I_l. Note that $-l < 0 < z_0 < z_1 < l$. To show that $s^- \leq s$ in I_l, we recall that $i_0 \leq i^+$ and $i_0 \geq 0$. Together with (2.12a), we deduce that

$$\delta s'' - cs' + \mu(\Lambda - s) - \beta s i^+ \leq 0 \text{ in } (-l, l). \tag{2.13}$$

Then (2.8) and (2.13) imply that the function $w_1 := s - s^-$ satisfies $\delta w_1'' - cw_1' - (\mu + \beta i^+) w_1 \leq 0$ in (z_0, l). In addition, from (2.12c) and the fact $s(z_0) > 0$ and $s^-(z_0) = 0$, we know that $w_1(z_0) > 0$ and $w_1(l) = s^+(l) - s^-(l) > 0$. Hence the maximum principle asserts that $w_1 \geq 0$ in $[z_0, l]$, which implies that $s^- \leq s$ in $[z_0, l]$. Together with the fact that $s^- \equiv 0 \leq s$ in $[-l, z_0]$, we get $s^- \leq s$ in $[-l, l]$. Now we show that $s \leq s^+$ in I_l. Recalling that $s^+ \equiv \Lambda$ and noting that $\gamma < \beta\Lambda$, one can easily see that s^+ satisfies

$$\delta(s^+)'' - c(s^+)' + \mu(\Lambda - s^+) - \beta s^+ i_0 + \gamma i_0 \leq 0 \quad \text{in } (-l, l).$$

Since $s^+(\pm l) = s(\pm l)$, we can use a similar argument as the proof for $s^- \leq s$ in $[z_0, l]$ to get that $s \leq s^+$ in I_l.

Finally, we claim that $i^- \leq i \leq i^+$ on I_l. Since

$$s^- i^- \leq s_0 i_0 \leq \Lambda i^+,$$

so that

$$i'' - ci' + \beta s^- i^- - (\gamma + \mu + \kappa) i \leq 0 \tag{2.14}$$

and

$$i'' - ci' + \beta \Lambda i^+ - (\gamma + \mu + \kappa) i \geq 0 \tag{2.15}$$

for all z in $(-l, l)$. Now we consider the function $w_2 = i - i^-$. From (2.12c) and the fact $i(z_1) > 0$ and $i^-(z_1) = 0$, we know that $w_2(z_1) > 0$ and $w_2(l) = i^+(l) - i^-(l) > 0$. In addition, (2.9) and (2.14) give that $w_2''(z) - cw_2'(z) - (\gamma + \mu + \kappa) w_2(z) \leq 0$ for all $z \in (z_1, l)$. Then it follows from the maximum principle that $w_2 \geq 0$ in $[z_1, l]$. This implies that $i^- \leq i$ in $[z_1, l]$. Together with the fact that $i^- \equiv 0 \leq i$ in $[-l, z_1]$, we get $i^- \leq i$ in I_l. Similarly, noting that $i^+(\pm l) = i(\pm l)$, one can easily use (2.15) and the maximum principle to deduce that $i \leq i^+$ in I_l. Hence the proof of this lemma is completed. □

Lemma 2.6 *$\mathcal{F}$ is a continuous mapping.*

Proof. For given (s_0, i_0) and $(\tilde{s_0}, \tilde{i_0})$ in E, let

$$(s, i) = \mathcal{F}(s_0, i_0) \text{ and } (\tilde{s}, \tilde{i}) = \mathcal{F}(\tilde{s_0}, \tilde{i_0}). \tag{2.16}$$

Consider the function $w_1 := s - \tilde{s}$. With a straightforward computation, one can verify that $w_1(-l) = w_1(l) = 0$ and

$$w_1'' - \frac{c}{\delta} w_1' + f_1(z) w_1 = h_1(z),$$

where

$$f_1(z) = -(\mu + \beta i_0(z))/\delta \text{ and } h_1(z) = \frac{1}{\delta}(\beta \tilde{s}(z) - \gamma)\left(i_0(z) - \tilde{i_0}(z)\right).$$

Since $0 \leq i_0 \leq i^+ \leq \|i^+\|_{C(I_l)} = e^{\lambda l}$ and $0 \leq \tilde{s} \leq s^+ \equiv \Lambda$, it follows that

$$-C_1 \leq f_1 \leq 0 \text{ and } |h_1| \leq C_2 \cdot \|i_0 - \tilde{i_0}\|_{C(I_l)},$$

where $C_1 := (\mu + \beta e^{\lambda l})/\delta$, $C_2 := (\beta\Lambda + \gamma)/\delta$. Then from Lemma A.1 in the appendix, it follows that there exists a positive constant C_3, depending only on C_1, δ, c, and l, such that

$$\|w_1\|_{C(I_l)} \leq C_2 C_3 \cdot \|i_0 - \tilde{i_0}\|_{C(I_l)},$$

which, together with definition of w_1, implies that

$$\|s - \tilde{s}\|_{C(I_l)} \leq C_2 C_3 \cdot \|i_0 - \tilde{i_0}\|_{C(I_l)}. \tag{2.17}$$

Next, consider the function $w_2 = i - \tilde{i}$. Again, with a straightforward computation, it follows that w_2 satisfies $w_2(-l) = w_2(l) = 0$ and

$$w_2'' - cw_2' - (\gamma + \mu + \kappa)w_2 = h_2(z),$$

where

$$h_2 = \beta\tilde{i_0}(\tilde{s_0} - s_0) + \beta s_0(\tilde{i_0} - i_0). \tag{2.18}$$

Since $0 \leq \tilde{i_0} \leq \|i^+\|_{C(I_l)} = e^{\lambda l}$ and $0 \leq s_0 \leq \Lambda$, we deduce from (2.18) that

$$|h_2| \leq \beta e^{\lambda l}\|s_0 - \tilde{s_0}\|_{C(I_l)} + \beta\Lambda\|i_0 - \tilde{i_0}\|_{C(I_l)}.$$

Then Lemma A.1 in the appendix asserts that there exists a positive constant C_4, depending only on γ, μ, κ, β, c, Λ, λ, and l, such that

$$\|w_2\|_{C(I_l)} \leq C_4 \left(\|s_0 - \tilde{s_0}\|_{C(I_l)} + \|i_0 - \tilde{i_0}\|_{C(I_l)}\right),$$

which, together with definition of w_2, implies that

$$\|i - \tilde{i}\|_{C(I_l)} \leq C_4 \left(\|s_0 - \tilde{s_0}\|_{C(I_l)} + \|i_0 - \tilde{i_0}\|_{C(I_l)}\right). \tag{2.19}$$

Finally, we use (2.16), (2.17), (2.19), and definition of the norm $\|\cdot\|_X$ to deduce that

$$\begin{aligned} & \|\mathcal{F}(s_0, i_0) - \mathcal{F}(\tilde{s_0}, \tilde{i_0})\|_X \\ = \; & \|(s, i) - (\tilde{s}, \tilde{i})\|_X \\ = \; & \|s - \tilde{s}\|_{C(I_l)} + \|i - \tilde{i}\|_{C(I_l)} \\ \leq \; & C_5(\|s_0 - \tilde{s_0}\|_{C(I_l)} + \|i_0 - \tilde{i_0}\|_{C(I_l)}) \\ = \; & C_5\|(s_0, i_0) - (\tilde{s_0}, \tilde{i_0})\|_X, \end{aligned} \tag{2.20}$$

where $C_5 = C_2C_3 + C_4$. Thus, for a given $\epsilon > 0$, we choose $0 < \sigma_1 < \epsilon/C_5$. Then, by (2.20), we have

$$\|\mathcal{F}(s_0, i_0) - \mathcal{F}(\tilde{s_0}, \tilde{i_0})\|_X < \epsilon,$$

for any (s_0, i_0), $(\tilde{s_0}, \tilde{i_0}) \in E$ such that $\|(s_0, i_0) - (\tilde{s_0}, \tilde{i_0})\|_X < \sigma_1$. This implies that $\mathcal{F}$ is a continuous mapping, thereby completing the proof of this lemma. ∎

Lemma 2.7 *$\mathcal{F}$ is precompact.*

Proof. The proof of this lemma are standard. We follows the proof of [7, Lemma 2.7].

For a given sequence $\{(s_{0,n}, i_{0,n})\}_{n\in\mathbb{N}}$ in E, let $(s_n, i_n) = \mathcal{F}(s_{0,n}, i_{0,n})$. Then Lemma 2.5 yields that $(s_n, i_n) \in E$. Since $0 \le s^- \le s^+ \equiv \Lambda$ and $0 \le i^- \le i^+ \le e^{\lambda l}$ in I_l, it follows from definition of the set E that the sequences

$$\{s_{0,n}\}, \{i_{0,n}\}, \{s_n\}, \{i_n\}, \{s_{0,n}i_{0,n}\}, \text{ and } \{s_n i_{0,n}\}$$

are uniformly bounded in I_l. Then, in view of Lemma A.2 in the appendix, we have that the sequences

$$\{s_n'\} \text{ and } \{i_n'\},$$

are also uniformly bounded in I_l. Hence Arzela-Ascoli theorem asserts that there exists a subsequence $\{(s_{n_j}, i_{n_j})\}$ of $\{(s_n, i_n)\}$ such that

$$(s_{n_j}, i_{n_j}) \to (s, i),$$

uniformly in I_l as $j \to \infty$, for some $(s, i) \in E$. This implies that the set $\overline{\mathcal{F}(E)}$ is compact in E, and hence that $\mathcal{F}$ is precompact. This establishes the assertion of this lemma. □

Finally, with the aid of Lemma 2.5-Lemma 2.7, we can apply the Schauder fixed point theorem to conclude that $\mathcal{F}$ has a fixed point (s_l, i_l), which is a nonnegative solution of system (2.10)-(2.11) satisfying $0 \le s^- \le s_l \le s^+ \equiv \Lambda$ and $0 \le i^- \le i_l \le i^+$ on I_l. Indeed, $s_l > \gamma/\beta$ on I_l. To see this, we note that $\underline{s} := \gamma/\beta$ satisfies

$$\delta \underline{s}'' - c\underline{s}' + \mu(\Lambda - \underline{s}) - \beta \underline{s} i + \gamma i \ge 0 \quad \text{in } (-l, l), \tag{2.21}$$

where we have used the fact $\Lambda > \gamma/\beta$. Let $w := s_l - \underline{s}$. Then, using (2.10a) and (2.21), we deduce that

$$\delta w'' - cw' - (\mu + \beta i)w \le 0 \quad \text{in } (-l, l).$$

In addition, since $\Lambda > \gamma/\beta$, it follows from (2.11) that $w(\pm l) = s_l(\pm l) - \underline{s}(\pm l) = \Lambda - \gamma/\beta > 0$. Then it follows from the maximum principle that $w > 0$ and so $s_l > \underline{s}$ on I_l. Hence $s_l > \gamma/\beta$ on I_l. From the above discussion, we have the following existence result for the truncated problem (2.10)-(2.11).

Lemma 2.8 *System (2.10)-(2.11) admits a solution (s_l, i_l) on I_l. Moreover,*

$$\gamma/\beta \le \max\{\gamma/\beta, s^-\} \le s_l \le s^+ \equiv \Lambda \text{ and } 0 \le i^- \le i_l \le i^+ \tag{2.22}$$

on I_l.

2.4 The construction of a candidate of traveling waves

In this subsection, we use the solution (s_l, i_l) of the truncated problem (2.10)-(2.11) and the limiting argument to obtain a solution (s, i) of system (1.2) satisfying $(s, i)(-\infty) = (\Lambda, 0)$. Hence if we could show that $(s, i)(+\infty) = (s^*, i^*)$, then (s, i) must be a traveling wave of system (1.1). Thus this observation would suggest that (s, i) is a good candidate of traveling wave solutions of system (1.1). The condition that $(s, i)(+\infty) = (s^*, i^*)$ will be verified in Sec. 3. Now we have the following lemma.

Lemma 2.9 *If $c > c_{min}$, then system (1.2) admits a solution (s,i) on $\mathbb{R}$ satisfying $\gamma/\beta < s < \Lambda$ and $i > 0$ over $\mathbb{R}$, $i(z) = \mathcal{O}(e^{\lambda z})$ as $z \to -\infty$, where λ is given by (1.4), and*

$$(s,i)(-\infty) = (\Lambda, 0) \text{ and } (s', i')(-\infty) = (0,0).$$

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ be an increasing sequence in (z_1, ∞) such that $l_n \to \infty$ as $n \to \infty$ and let (s_n, i_n), $n \in \mathbb{N}$, be a solution of system (2.10)-(2.11) with $l = l_n$. For any fixed $N \in \mathbb{N}$, since the function i^+ is bounded above in $[-l_N, l_N]$, it follows from (2.22) that the sequences

$$\{s_n\}_{n\geq N}, \{i_n\}_{n\geq N}, \text{ and } \{s_n i_n\}_{n\geq N}$$

are uniformly bounded in $[-l_N, l_N]$. Then we can use Lemma A.2 to infer that the sequences

$$\{s_n'\}_{n\geq N} \text{ and } \{i_n'\}_{n\geq N}$$

are also uniformly bounded in $[-l_N, l_N]$. Using (2.10), we can express s_n'' and i_n'' in terms of s_n, i_n, s_n' and i_n'. Differentiating (2.10), we can use the resulting equations to express s_n''' and i_n''' in terms of s_n, i_n, s_n', i_n', s_n'' and i_n''. Consequently, the sequences

$$\{s_n''\}_{n\geq N}, \{i_n''\}_{n\geq N}, \{s_n'''\}_{n\geq N} \text{ and } \{i_n'''\}_{n\geq N}$$

are uniformly bounded in $[-l_N, l_N]$. With the aid of Arzela-Ascoli theorem, we can use a diagonal process to get a subsequence $\{(s_{n_j}, i_{n_j})\}$ of $\{(s_n, i_n)\}$ such that

$$s_{n_j} \to s, s_{n_j}' \to s', s_{n_j}'' \to s'',$$

and

$$i_{n_j} \to i, i_{n_j}' \to i', i_{n_j}'' \to i'',$$

uniformly in any compact interval of $\mathbb{R}$ as $n \to \infty$, for some functions s and i in $C^2(\mathbb{R})$. Then it is easy to see that (s,i) is a nonnegative solution of system (1.2) and satisfies

$$\gamma/\beta \leq \max\{\gamma/\beta, s^-\} \leq s \leq s^+ \equiv \Lambda \text{ and } 0 \leq i^- \leq i \leq i^+ \tag{2.23}$$

over $\mathbb{R}$. From definitions of s^- and i^+, we see that $s^-(z) \to \Lambda$ and $i^+(z) \to 0$ as $z \to -\infty$. This, together with (2.23), implies that

$$(s,i)(-\infty) = (\Lambda, 0), \tag{2.24}$$

and $i(z) = \mathcal{O}(e^{\lambda z})$ as $z \to -\infty$, where λ is given by (1.4).

Furthermore, we claim that $\gamma/\beta < s < \Lambda$ and $i > 0$ over $\mathbb{R}$, and

$$(s', i')(-\infty) = (0,0). \tag{2.25}$$

For contradiction, we assume that $i(\tilde{z}_1) = 0$ for some $\tilde{z}_1 \in \mathbb{R}$. Then $i'(\tilde{z}_1) = 0$. Therefore the uniqueness gives that $i \equiv 0$, which contradicts the fact that $i \geq i^- > 0$ on $(-\infty, z_1)$. Hence $i > 0$ over $\mathbb{R}$. To prove $s < \Lambda$ over $\mathbb{R}$, we also use a contradictory argument and assume that $s(\tilde{z}_2) = \Lambda$ for some $\tilde{z}_2 \in \mathbb{R}$. In this case, $s'(\tilde{z}_2) = 0$ and $s''(\tilde{z}_2) \leq 0$. This contradicts (1.2a) with $z = \tilde{z}_2$. Hence $s < \Lambda$ over $\mathbb{R}$. Suppose $s(\tilde{z}_3) = \gamma/\beta$ for some $\tilde{z}_3 \in \mathbb{R}$, then $s'(\tilde{z}_3) = 0$ and $s''(\tilde{z}_3) \geq 0$. This contradicts (1.2a) with $z = \tilde{z}_3$. Hence $s > \gamma/\beta$ over $\mathbb{R}$.

To prove (2.25), we use Eq. (1.2a) to deduce that

$$s'(z) = e^{-\frac{c}{\delta}(z-\xi)} s'(\xi) - \frac{1}{\delta} e^{-\frac{c}{\delta}z} \int_{\xi}^{z} e^{\frac{c}{\delta}\tau} \big(\mu(\Lambda - s(\tau)) - \beta s(\tau) i(\tau) + \gamma i(\tau)\big) d\tau. \tag{2.26}$$

By fixing ξ and letting $z \to -\infty$ in the equality (2.26), we immediately deduces that,

$$\begin{aligned} \limsup_{z\to-\infty} |s'(z)| &\leq \frac{1}{\delta} \max_{\tau\geq\xi} |\mu(\Lambda - s(\tau)) - \beta s(\tau) i(\tau) + \gamma i(\tau)| \cdot \limsup_{z\to-\infty} e^{-\frac{c}{\delta}z} \int_{\xi}^{z} e^{\frac{c}{\delta}\tau} d\tau \\ &\leq \frac{1}{c} \max_{\tau\geq\xi} |\mu(\Lambda - s(\tau)) - \beta s(\tau) i(\tau) + \gamma i(\tau)| \end{aligned}$$

for $s \in \mathbb{R}$. Together with the fact that $\mu(\Lambda - s(-\infty)) - \beta s(-\infty) i(-\infty) + \gamma i(-\infty) = 0$, we can deduce that $s'(-\infty) = 0$. Similarly, using equation (1.2b) and arguing as above, we also get $i'(-\infty) = 0$. □

3 Existence of non-critical waves of system (1.1)

Throughout this section, we always assume that $c > c_{min}$. Now we will establish the assertion of Theorem 1.1 (II), which is restated in the following lemma for the convenience of the readers.

Lemma 3.1 *If $c > c_{min}$, then system (1.2)-(1.3) admits a nonnegative solution (s, i) with the following properties:*

- (i) $\gamma/\beta < s < \Lambda$ *and* $i > 0$ *over* $\mathbb{R}$.
- (ii) *There exists a* $\gamma^* > 0$ *such that there hold*
  - (a) *if* $\gamma \in (0, \gamma^*)$*, then the solution* (s, i) *approaches* (s^*, i^*) *monotonically for large* z.
  - (b) *if* $\gamma > \gamma^*$*, then the solution* (s, i) *has exponentially damped oscillations about* (s^*, i^*) *for large* z.
- (iii) $i(z) = \mathcal{O}(e^{\lambda z})$ *as* $z \to -\infty$*, where* λ *is given by (1.4).*

- In the remaining part of this section, we will prove Lemma 3.1. Recall from Sec. 2.4 that a good candidate for the solution of system (1.2)-(1.3) is the one given in Lemma 2.9 which will be denoted by (s, i). Lemma 2.9 indicates that (s, i) satisfies $(s, i)(-\infty) = (\Lambda, 0)$ and the assertions (i) and (iii) of Lemma 3.1. Further, if $(s, i)(\infty) = (s^*, i^*)$, then a straightforward eigenvalue analysis of system (1.2)-(1.3) around the equilibrium point (s^*, i^*) shows that the assertions (ii) of Lemma 3.1 holds for (s, i). Hence, in order to complete the proof of Lemma 3.1, it remains to verify that (s, i) satisfies

$$(s, i)(\infty) = (s^*, i^*). \tag{3.1}$$

To show the equality (3.1), we set

$$\hat{\mathbf{u}}(z) = s(z) \text{ and } \hat{\mathbf{v}}(z) = i(z). \tag{3.2}$$

Then the governing equation for $(\mathsf{u}, \mathsf{v}) = (\hat{\mathbf{u}}, \hat{\mathbf{v}})$ is given by

$$\delta \mathsf{u}'' - c\mathsf{u}' = \mu(\mathsf{u} - \Lambda) + \beta \mathsf{u}\mathsf{v} - \gamma \mathsf{v}, \tag{3.3a}$$
$$\mathsf{v}'' - c\mathsf{v}' = -\beta \mathsf{u}\mathsf{v} + (\gamma + \mu + \kappa)\mathsf{v}, \tag{3.3b}$$

where the prime denotes the differentiation with respect to z, and (3.1) becomes the equality

$$(\hat{\mathbf{u}}, \hat{\mathbf{v}})(\infty) = (s^*, i^*). \tag{3.4}$$

Further, in view of the definition of $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ and Lemma 2.9, we have that $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ is a solution of (3.3) on $\mathbb{R}$ satisfying

$$\gamma/\beta < \hat{\mathbf{u}} < \Lambda \text{ and } \hat{\mathbf{v}} > 0 \tag{3.5}$$

over $\mathbb{R}$, and

$$(\hat{\mathbf{u}}, \hat{\mathbf{v}})(-\infty) = (\Lambda, 0) \text{ and } (\hat{\mathbf{u}}', \hat{\mathbf{v}}')(-\infty) = (0, 0).$$

We will keep the notation $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ *throughout the remaining of this section.*

In order to show the equality (3.4) (i.e., $(s, i)(\infty) = (s^*, i^*)$, we write (3.3) as a system of first-order ODEs:

$$\mathsf{u}' = \mathsf{w}, \tag{3.6a}$$
$$\delta \mathsf{w}' = c\mathsf{w} + \mu(\mathsf{u} - \Lambda) + \beta \mathsf{u}\mathsf{v} - \gamma \mathsf{v}, \tag{3.6b}$$
$$\mathsf{v}' = \mathsf{y}, \tag{3.6c}$$
$$\mathsf{y}' = c\mathsf{y} - \beta \mathsf{u}\mathsf{v} + (\gamma + \mu + \kappa)\mathsf{v}. \tag{3.6d}$$

Next we borrow the idea of [4] to define the Lyapunov function $\mathcal{L}$ by

$$\begin{aligned} & \mathcal{L}(\mathsf{u}, \mathsf{w}, \mathsf{v}, \mathsf{y}) \\ := & -\left(\delta \mathsf{w} - c\mathsf{u} - \delta(\mu + \kappa)\frac{\mathsf{w}}{\beta \mathsf{u} - \gamma} + \frac{c(\mu + \kappa)}{\beta} \ln \frac{\beta \mathsf{u} - \gamma}{\beta s^* - \gamma}\right) - \left(\mathsf{y} - c\mathsf{v} - i^* \frac{\mathsf{y}}{\mathsf{v}} + c i^* \ln \frac{\mathsf{v}}{i^*}\right) \\ = & \ \mathcal{L}_1(\mathsf{u}, \mathsf{w}, \mathsf{v}, \mathsf{y}) + \mathcal{L}_2(\mathsf{u}, \mathsf{w}, \mathsf{v}, \mathsf{y}). \end{aligned} \tag{3.7}$$

With the use of a straightforward computation, the orbital derivative of $\mathcal{L}$ along the solution $\chi(z) := (\hat{\mathbf{u}}(z), \hat{\mathbf{w}}(z), \hat{\mathbf{v}}(z), \hat{\mathbf{y}}(z))$, where $\hat{\mathbf{w}} := \hat{\mathbf{u}}'(z)$ and $\hat{\mathbf{y}} := \hat{\mathbf{v}}'(z)$, of system (3.6) is

$$\begin{aligned} & \frac{d}{dz}\mathcal{L}(\chi(z)) \\ = & \ \nabla \mathcal{L}(\chi(z)) \cdot \chi'(z) \\ = & -\delta\beta(\mu + \kappa)\frac{\hat{\mathbf{w}}(z)^2}{(\beta\hat{\mathbf{u}}(z) - \gamma)^2} - i^* \frac{\hat{\mathbf{y}}(z)^2}{\hat{\mathbf{v}}(z)^2} - \frac{\mu\beta(\beta\Lambda - \gamma)(s^* - \hat{\mathbf{u}}(z))^2}{(\mu + \kappa)(\beta\hat{\mathbf{u}} - \gamma)}, \end{aligned}$$

which, together with the fact that $\beta\Lambda - \gamma > 0$ and $\beta\hat{\mathbf{u}} - \gamma > 0$, yields

$$\frac{d}{dz}\mathcal{L}(\chi(z)) \leq 0.$$

Hence the orbital derivative of $\mathcal{L}$ along $\chi(z)$ is non-positive, and

$$\mathcal{L}(\chi(z)) \leq \mathcal{L}(\chi(0)), \forall z \geq 0. \tag{3.8}$$

Now we collect the estimates of $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ and its derivatives in the following lemma whose proof is deferred to the appendix B.

Lemma 3.2 (i) *There exists a positive constant B such that $0 < \hat{\mathbf{v}}(z) < B$ for all $z \in \mathbb{R}$.*

(ii) *There exist positive constants L_i, $i = 1, 2$, such that*

$$-L_1(\beta\hat{\mathbf{u}}(z) - \gamma) < \hat{\mathbf{u}}'(z) < L_2(\beta\hat{\mathbf{u}}(z) - \gamma)$$

for all $z \geq 0$.

(iii)

$$-\frac{\mu + \gamma + \kappa}{c} \cdot \hat{\mathbf{v}}(z) \leq \hat{\mathbf{v}}'(z) \leq \frac{c}{2}\hat{\mathbf{v}}(z) \ \forall z \in \mathbb{R}.$$

In the sequel, we retain the notation B, and L_i, $i = 1, 2$.

Note that (3.5) and Lemma 3.2 assert that the solution $\chi(z) = (\hat{\mathbf{u}}(z), \hat{\mathbf{w}}(z), \hat{\mathbf{v}}(z), \hat{\mathbf{y}}(z))$ with $(\hat{\mathbf{w}}, \hat{\mathbf{y}}) = (\hat{\mathbf{u}}', \hat{\mathbf{v}}')$ of system (3.6) is positively invariant in the open bounded set $\mathcal{D}$ for all $z \geq 0$, where $\mathcal{D}$ is defined by

$$\mathcal{D} := \left\{ (\mathsf{u}, \mathsf{w}, \mathsf{v}, \mathsf{y}) | \frac{\gamma}{\beta} < \mathsf{u} < \Lambda, 0 < \mathsf{v} < B, -L_1(\beta\mathsf{u} - \gamma) < \mathsf{w} < L_2(\beta\mathsf{u} - \gamma), -\tfrac{2(\mu+\gamma+\kappa)}{c}\mathsf{v} < \mathsf{y} < c\mathsf{v} \right\}.$$

On the other hand, one can easily see that $\mathcal{L}$ is continuous, and, by (3.5) and Lemma 3.2, that $\mathcal{L}$ is bounded below on $\mathcal{D}$. Taken together, it follows from LaSalle's invariance principle that $\chi(z) \to (s^*, 0, i^*, 0)$ as $z \to \infty$, and so $(s, i)(\infty) = (\hat{\mathbf{u}}, \hat{\mathbf{v}})(\infty) = (s^*, i^*)$. This completes the proof of Lemma 3.1, and hence the proof of Theorem 1.1. □

Appendix A

In this appendix, we collect some *a priori* estimates in [6] for solutions of the inhomogeneous linear equation

$$w''(z) + Aw'(z) + f(z)w(z) = h(z). \tag{A.1}$$

Lemma A.1 (Lemma 3.2 of [6])

Let A be a positive constant and let f and h be continuous functions on $[a, b]$. Suppose that $w \in C([a, b]) \cap C^2((a, b))$ satisfies the differential equation (A.1) in (a, b) and $w(a) = w(b) = 0$. If

$$-C_1 \leq f \leq 0 \ \text{and} \ |h| \leq C_2 \ \text{on} \ [a, b],$$

for some constants C_1, C_2, then there exists a positive constant C_3, depending only on A, C_1, and the length of the interval $[a, b]$, such that

$$\|w\|_{C([a,b])} \leq C_2 C_3.$$

Lemma A.2 (Lemma 3.3 of [6])

Let A, f, and h be as in Lemma A.1. Suppose that $w \in C([a,b]) \cap C^2((a,b))$ satisfies (A.1) in (a,b). If $\|w\|_{C([a,b])} \leq C_0$ for some constant C_0, then there exists a positive constant C_4, depending only on A, C_0, C_1, C_2, and the length of the interval $[a,b]$, such that

$$\|w'\|_{C([a,b])} \leq C_4.$$

Appendix B

In this appendix, we will show the *a prior* estimates for $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ given in Lemma 3.2. Since the proof of these *a prior* estimates are similar to those in our previous work [7], we will only sketch the necessary ingredients and refer the readers to [7] for further details.

B.1 Estimates of the derivative of $\hat{\mathbf{v}}$

We first derive the estimate for the derivative of $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$. This will completes the proof of Lemma 3.2 (iii). Recall that $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ is defined by (3.2).

Lemma B.1 *For each $z \in \mathbb{R}$, the following inequalities hold:*

$$\hat{\mathbf{v}}'(z) \leq \frac{c}{2}\hat{\mathbf{v}}(z), \tag{B.1}$$

$$\hat{\mathbf{v}}'(z) \geq -\frac{\mu+\gamma+\kappa}{c} \cdot \hat{\mathbf{v}}(z), \tag{B.2}$$

$$\hat{\mathbf{u}}'(z) \leq \mu\Lambda/c. \tag{B.3}$$

Proof. The proof of the inequalities (B.1) and (B.3) follows the line of the inequalities (3.9) and (3.11) in [7, Lemma 3.2].

For the proof of (B.2), we set

$$\Phi(z) := c\hat{\mathbf{v}}'(z) + (\mu+\gamma+\kappa)\hat{\mathbf{v}}(z).$$

Then the proof follows the line of the inequality (3.10) in [7, Lemma 3.2]. Hence the proof of this lemma is completed. □

B.2 Boundedness of $\hat{\mathbf{v}}$

In this subsection, we wii prove that $\hat{\mathbf{v}}$ is bounded over $\mathbb{R}$. This will completes the proof of Lemma 3.2 (i). Recal that $\lim_{z\to-\infty}\hat{\mathbf{v}}(z) = 0$. For contradiction, we assume that $\limsup_{z\to\infty}\hat{\mathbf{v}}(z) = \infty$. Then this gives rise to two possibilities: (i) $\lim_{z\to\infty}\hat{\mathbf{v}}(z) = \infty$; or (ii) $\liminf_{z\to\infty}\hat{\mathbf{v}}(z) < \limsup_{z\to\infty}\hat{\mathbf{v}}(z) = \infty$. In this subsection, we will exclude these two possibilities.

B.2.1 The case that $\lim_{z\to\infty} \hat{\mathbf{v}}(z) = \infty$

In this subsection, we will exclude the possibility that $\lim_{z\to\infty} \hat{\mathbf{v}}(z) = \infty$. Specifically, we state it in the following lemma.

Lemma B.2 *The solution* $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ *cannot satisfy* $\lim_{z\to\infty} \hat{\mathbf{v}}(z) = \infty$.

Proof. Set

$$\Psi := \delta\hat{\mathbf{u}}' - c\hat{\mathbf{u}} + \hat{\mathbf{v}}' - c\hat{\mathbf{v}}$$

for $z \in \mathbb{R}$. Then the proof follows the line of the proof in [7, Lemma 3.3]. □

B.2.2 The case that $\liminf_{z\to\infty} \hat{\mathbf{v}}(z) < \limsup_{z\to\infty} \hat{\mathbf{v}}(z) = \infty$

In this subsection, we will exclude the case that $\liminf_{z\to\infty} \hat{\mathbf{v}}(z) < \limsup_{z\to\infty} \hat{\mathbf{v}}(z) = \infty$. For contradiction, we assume that $\liminf_{z\to\infty} \hat{\mathbf{v}}(z) < \limsup_{z\to\infty} \hat{\mathbf{v}}(z) = \infty$. Then $\hat{\mathbf{v}}(z)$ oscillates infinitely many times as $z \to \infty$. To derive a contradiction, we need five auxiliary lemmas (i.e., Lemma B.3 - Lemma B.7).

Lemma B.3 $\hat{\mathbf{u}}(z)$ *oscillates infinitely many times as* $z \to \infty$.

Proof. The proof follows the line of the proof in [7, Lemma 3.4]. □

Lemma B.4 $\hat{\mathbf{u}} \geq \gamma/\beta + \epsilon_0$ *on* $\mathbb{R}$ *for some positive constant* ϵ_0*. In the sequel, we retain the notation* ϵ_0.

Proof. The proof is a slight modification of that for [7, Lemma 3.5].

First, recall that $\hat{\mathbf{u}}(-\infty) = \Lambda > \gamma/\beta$, $\hat{\mathbf{u}}(z) > \gamma/\beta$ for all $z \in \mathbb{R}$, and that $\hat{\mathbf{u}}$ oscillates infinitely many times as $z \to \infty$. Therefore, if the assertion of the lemma is false, then there exists a sequence of positive numbers $\{z_n\} \to \infty$ such that $\hat{\mathbf{u}}$ has a local minimum at z_n and $\hat{\mathbf{u}}(z_n) \to \gamma/\beta$ as $n \to \infty$.

With the use of (B.1) and (B.2), we have that

$$\mathcal{L}_2(\chi(z)) \geq -\left(-\frac{c}{2}\hat{\mathbf{v}}(z) + \frac{(\mu+\gamma+\kappa)i^*}{c} + ci^* \ln\frac{\hat{\mathbf{v}}(z)}{i^*}\right) := \psi_3(\hat{\mathbf{v}}(z)).$$

Since $\psi_3(0+) = \infty$ and $\psi_3(\infty) = \infty$, the function ψ_3 is bounded below in $(0, \infty)$. Hence the above inequality implies that $\mathcal{L}_2(\chi(z))$ is bounded below for $z \geq 0$. Further, since $\hat{\mathbf{u}}(z_n)$ is a local minimum of $\hat{\mathbf{u}}$, we have that $\hat{\mathbf{w}}(z_n) = \hat{\mathbf{u}}'(z_n) = 0$, and so

$$\mathcal{L}_1(\chi(z_n)) = \left(c\hat{\mathbf{u}}(z_n) - \frac{c(\mu+\kappa)}{\beta}\ln\frac{\beta\hat{\mathbf{u}}(z_n) - \gamma}{\beta s^* - \gamma}\right) \to \infty \text{ as } n \to \infty,$$

where we have used $\beta\hat{\mathbf{u}}(z_n) \to \gamma$ as $n \to \infty$, and $\beta s^* - \gamma > 0$. Taken together, we conclude that $\mathcal{L}(\chi(z_n)) \to \infty$ as $n \to \infty$. This is a contradiction to the fact that $\mathcal{L}(\chi(z))$ is decreasing in z. Hence this completes the proof of this lemma. □

Lemma B.5 *There exists a $M_1 \geq i^*$ such that for $\hat{\mathbf{v}}(z) \geq M_1$ with $z \geq 0$, we have $\hat{\mathbf{u}}'(z) < 0$. In the sequel, we retain the notation M_1.*

Proof. The proof is a slight modification of that for [7, Lemma 3.6].

First, since $c\mathsf{v}/2 - ci^* \ln \mathsf{v} \to \infty$ as $\mathsf{v} \to \infty$, there exists a large $M_1 \geq i^*$ such that for $\mathsf{v} \geq M_1$, we have

$$\left(\frac{c}{2}\mathsf{v} - ci^* \ln \mathsf{v}\right) + ci^* \ln i^* > \frac{\delta\mu\Lambda}{c} + \frac{c(\mu+\kappa)}{\beta} \ln \frac{\beta\Lambda - \gamma}{\beta s^* - \gamma} + 2\left|\mathcal{L}(\chi(0))\right|. \tag{B.4}$$

Next, with the use of (B.1) and (B.3), we deduce that for all z with $\hat{\mathbf{w}}(z) \geq 0$, there holds

$$\mathcal{L}_1(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})(z) \geq -\frac{\delta\mu\Lambda}{c} - \frac{c(\mu+\kappa)}{\beta} \ln \frac{\beta\Lambda - \gamma}{\beta s^* - \gamma}, \tag{B.5}$$

and for all z with $\hat{\mathbf{y}}(z) \geq 0$, there holds

$$\mathcal{L}_2(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})(z) \geq \left(\frac{c}{2}\hat{\mathbf{v}}(z) - ci^* \ln \hat{\mathbf{v}}(z)\right) + ci^* \ln i^*. \tag{B.6}$$

Now, for z with $\hat{\mathbf{v}}(z) \geq i^*$ and $\hat{\mathbf{y}}(z) < 0$, we estimate $\mathcal{L}_2(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})$ as follows:

$$\begin{aligned}
\mathcal{L}_2(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})(z) &= -\hat{\mathbf{y}}(z) + c\hat{\mathbf{v}}(z) + i^* \frac{\hat{\mathbf{y}}(z)}{\hat{\mathbf{v}}(z)} - ci^* \ln \frac{\hat{\mathbf{v}}(z)}{i^*} \\
&= -\left(1 - \frac{i^*}{\hat{\mathbf{v}}(z)}\right)\hat{\mathbf{y}} + \left(c\hat{\mathbf{v}}(z) - ci^* \ln \hat{\mathbf{v}}(z)\right) + ci^* \ln i^* \\
&\geq \left(c\hat{\mathbf{v}}(z) - ci^* \ln \hat{\mathbf{v}}(z)\right) + ci^* \ln i^*.
\end{aligned} \tag{B.7}$$

We are now ready to establish the assertion of this lemma. For contradiction, suppose that there exists a $\hat{z}_1 \geq 0$ such that $\hat{\mathbf{v}}(\hat{z}_1) \geq M_1$ and $\hat{\mathbf{w}}(\hat{z}_1) = \hat{\mathbf{u}}'(\hat{z}_1) \geq 0$. Then with the aid of (B.5)-(B.7), it follows from the choice of M_1 that

$$\mathcal{L}(\chi(\hat{z}_1)) = \mathcal{L}(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})(\hat{z}_1) > 2\left|\mathcal{L}(\chi(0))\right|,$$

which contradicts (3.8). The proof of this lemma is thus completed. ∎

Lemma B.6 *Suppose that $\hat{\mathbf{v}}(\hat{z}_0) \geq M_1$ and $\hat{\mathbf{v}}'(\hat{z}_0) = 0$ for some $\hat{z}_0 \in \mathbb{R}$. Then $\hat{\mathbf{v}}(\hat{z}_0)$ cannot be a local minimum.*

Proof. The proof follows the line of the proof in [7, Lemma 3.7]. ∎

Lemma B.7 *There exist positive constants k_1 and $M_2 > \max\{M_1, \limsup_{z\to\infty} \hat{\mathbf{v}}(z)\}$ such that for $z \geq 0$ with $\hat{\mathbf{v}}(z) \geq M_2$, we have $\hat{\mathbf{v}}(z) \leq -k_1 \hat{\mathbf{u}}'(z)$. In the sequel, we retain the notations k_1 and M_2.*

Proof. The proof is a slight modification of that for [7, Lemma 3.8].

To begin with, we set

$$k_1 := \frac{4\delta(\mu+\kappa)}{\beta\epsilon_0 c}.$$

Since

$$\frac{c}{4}\mathsf{v} - ci^* \ln \frac{\mathsf{v}}{i^*} \to \infty \text{ as } \mathsf{v} \to \infty,$$

there exists a large $M_2 > M_1$ such that

$$\frac{c}{4}\hat{\mathbf{v}}(z) - ci^* \ln \frac{\hat{\mathbf{v}}(z)}{i^*} \geq \frac{c(\mu+\kappa)}{\beta} \ln \frac{\beta\Lambda - \gamma}{\beta s^* - \gamma} + |\mathcal{L}(\chi(0))|, \tag{B.8}$$

for all z with $\hat{\mathbf{v}}(z) \geq M_2$. Now we set $\mathcal{Z} := \{z \geq 0 : \hat{\mathbf{v}}(z) \geq M_2\}$.

Next we estimate $\mathcal{L}_1(\chi(z))$ for $z \in \mathcal{Z}$. From Lemma B.5 and Lemma B.4, we have $\hat{\mathbf{w}}(z) = \hat{\mathbf{u}}'(z) < 0$ and $\beta\hat{\mathbf{u}} - \gamma \geq \beta\epsilon_0$ for $z \in \mathcal{Z}$. Then the following inequality holds for $z \in \mathcal{Z}$,

$$\begin{aligned}\mathcal{L}_1(\chi(z)) &= -\delta\hat{\mathbf{w}}(z) + c\hat{\mathbf{u}}(z) + \delta(\mu+\kappa)\frac{\hat{\mathbf{w}}(z)}{(\beta\hat{\mathbf{u}}(z)-\gamma)} - \frac{c(\mu+\kappa)}{\beta}\ln\frac{\beta\hat{\mathbf{u}}(z)-\gamma}{\beta s^*-\gamma}, \\ &\geq \frac{\delta(\mu+\kappa)}{\beta\epsilon_0}\cdot\hat{\mathbf{w}}(z) - \frac{c(\mu+\kappa)}{\beta}\ln\frac{\beta\Lambda-\gamma}{\beta s^*-\gamma}.\end{aligned} \tag{B.9}$$

Now we turn to estimate $\mathcal{L}_2(\chi(z))$ for $z \in \mathcal{Z}$. Indeed, since $i^*/\hat{\mathbf{v}}(z) < 1$ for $z \in \mathcal{Z}$, we can use (B.1) to deduce that for $z \in \mathcal{Z}$, it holds

$$\begin{aligned}\mathcal{L}_2(\chi(z)) &= -\hat{\mathbf{y}}(z) + c\hat{\mathbf{v}}(z) + i^*\frac{\hat{\mathbf{y}}(z)}{\hat{\mathbf{v}}(z)} - ci^*\ln\frac{\hat{\mathbf{v}}(z)}{i^*} \\ &= \left[-\left(1-\frac{i^*}{\hat{\mathbf{v}}(z)}\right)\hat{\mathbf{y}}(z) + \frac{c}{2}\hat{\mathbf{v}}(z)\right] + \frac{c}{4}\hat{\mathbf{v}}(z) + \left(\frac{c}{4}\hat{\mathbf{v}}(z) - ci^*\ln\frac{\hat{\mathbf{v}}(z)}{i^*}\right) \\ &\geq \frac{c}{4}\hat{\mathbf{v}}(z) + \left(\frac{c}{4}\hat{\mathbf{v}}(z) - ci^*\ln\frac{\hat{\mathbf{v}}(z)}{i^*}\right) \qquad \text{(using (B.1))} \\ &\geq \frac{c}{4}\hat{\mathbf{v}}(z) + \left(\frac{c(\mu+\kappa)}{\beta}\ln\frac{\beta\Lambda-\gamma}{\beta s^*-\gamma} + |\mathcal{L}(\chi(0))|\right). \quad \text{(using (B.8))}\end{aligned} \tag{B.10}$$

In view of (B.9)-(B.10) and the definition of k_1, we have that for $z \in \mathcal{Z}$,

$$\begin{aligned}\mathcal{L}(\chi(z)) &= \mathcal{L}_1(\chi(z)) + \mathcal{L}_2(\chi(z)) \\ &\geq \frac{c}{4}\cdot\big(\hat{\mathbf{v}}(z) + k_1\hat{\mathbf{w}}(z)\big) + |\mathcal{L}(\chi(0))|.\end{aligned}$$

Rearranging the above inequality, we deduce that

$$\frac{c}{4}\cdot\big(\hat{\mathbf{v}}(z) + k_1\hat{\mathbf{w}}(z)\big) \leq \mathcal{L}(\chi(z)) - |\mathcal{L}(\chi(0))| \leq 0,$$

which, together with $\hat{\mathbf{w}}(z) = \hat{\mathbf{u}}'(z)$, yields $\hat{\mathbf{v}}(z) \leq -k_1\hat{\mathbf{u}}'(z)$. Hence the proof is completed. □

Now we are ready to exclude the case that $\liminf_{z\to\infty}\hat{\mathbf{v}}(z) < \limsup_{z\to\infty}\hat{\mathbf{v}}(z) = \infty$.

Lemma B.8 *The solution* $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ *cannot satisfy the inequality* $\liminf_{z\to\infty}\hat{\mathbf{v}}(z) < \limsup_{z\to\infty}\hat{\mathbf{v}}(z) = \infty$*.*

Proof. For contradiction, we assume that $\liminf_{z\to\infty} \hat{\mathbf{v}}(z) < \limsup_{z\to\infty} \hat{\mathbf{v}}(z) = \infty$. Then, in view of Lemma B.5 and Lemma B.6, we can choose positive numbers $\hat{z}_0$ and $\hat{z}_1$ such that $\hat{\mathbf{v}}(\hat{z}_0) = M_2$, $\hat{\mathbf{v}}'(z) \geq 0$ for $z \in [\hat{z}_0, \hat{z}_1)$, $\hat{\mathbf{v}}'(\hat{z}_1) = 0$, $\hat{\mathbf{u}}'(z) < 0$ for $z \in [\hat{z}_0, \hat{z}_1]$, and

$$c\hat{\mathbf{v}}(\hat{z}_1) > cM_2 + k_1 \int_0^{\Lambda} |\beta\hat{\mathbf{u}} - \gamma - \mu - \kappa| d\hat{\mathbf{u}}. \tag{B.11}$$

Hence $[\hat{z}_0, \hat{z}_1] \subset \mathscr{Z} := \big\{z \geq 0 : \hat{\mathbf{v}}(z) \geq M_2\big\}$.

Then the remaining of the proof is a slight modification of that for [7, Lemma 3.9]. Hence the proof of this lemma is completed. □

B.3 Estimate of the derivative of $\hat{\mathbf{u}}$

In the following lemma, we derive the estimate for the derivative of $\hat{\mathbf{u}}$. This will completes the proof of Lemma 3.2 (ii). For this, we recall from Sec. B.2 that $\hat{\mathbf{v}}$ is bounded over $\mathbb{R}$. Thus we can choose a positive constant B such that $\hat{\mathbf{v}}(z) < B$ for all $z \in \mathbb{R}$.

Lemma B.9 *There exist positive constants L_i, $i = 1, 2$, such that*

$$-L_1(\beta\hat{\mathbf{u}}(z) - \gamma) < \hat{\mathbf{u}}'(z) < L_2(\beta\hat{\mathbf{u}}(z) - \gamma) \tag{B.12}$$

for all $z \geq 0$.

Proof. (1) We show that $-L_1(\beta\hat{\mathbf{u}}(z) - \gamma) < \hat{\mathbf{u}}'(z)$ for all $z \geq 0$, if L_1 is a sufficiently large constant such that $-L_1(\beta\hat{\mathbf{u}}(0) - \gamma) < \hat{\mathbf{u}}'(0)$ and $L_1 \geq 2B/c$.

Let

$$\Phi_1(z) := \hat{\mathbf{u}}'(z) + L_1(\beta\hat{\mathbf{u}}(z) - \gamma).$$

It suffices to show that $\Phi_1(z) > 0$ for all $z \geq 0$. Note that $\Phi_1(0) > 0$. For contradiction, we assume that there exists $\hat{z}_1 > 0$ such that $\Phi_1(\hat{z}_1) = 0$ and $\Phi_1'(\hat{z}_1) \leq 0$. Then there are two possibilities: either

$$\Phi_1(z) \leq 0, \forall z \geq \hat{z}_1 \tag{B.13}$$

or

$$\Phi_1(\hat{z}_2) = 0 \text{ and } \Phi_1'(\hat{z}_2) \geq 0, \tag{B.14}$$

for some $\hat{z}_2 \geq \hat{z}_1$. For the first case, (B.13) gives

$$c\hat{\mathbf{u}}'(z) \leq -2B(\beta\hat{\mathbf{u}}(z) - \gamma), \forall z \geq \hat{z}_1.$$

Together with the fact that $0 \leq \hat{\mathbf{v}} \leq B$, $\beta\hat{\mathbf{u}} - \gamma > 0$, and $\hat{\mathbf{u}} < \Lambda$, we deduce from (3.3a) that

$$\delta\hat{\mathbf{u}}'' = c\hat{\mathbf{u}}' + (\beta\hat{\mathbf{u}} - \gamma)\hat{\mathbf{v}} - \mu(\Lambda - \hat{\mathbf{u}}) \leq -B(\beta\hat{\mathbf{u}} - \gamma) < 0, \forall z \geq \hat{z}_1,$$

which implies that $\hat{\mathbf{u}}'$ is decreasing in $[\hat{z}_1, \infty)$. Hence $\hat{\mathbf{u}}'(z) \leq \hat{\mathbf{u}}'(\hat{z}_1) = -L_1(\beta\hat{\mathbf{u}}(\hat{z}_1) - \gamma) < 0$ for all $z \geq \hat{z}_1$, which contradicts the boundedness of $\hat{\mathbf{u}}$. For the second case, (B.14) yields that

$$\hat{\mathbf{u}}'(\hat{z}_2) = -L_1(\beta\hat{\mathbf{u}}(\hat{z}_2) - \gamma) \tag{B.15}$$

and

$$\hat{\mathbf{u}}''(\hat{z}_2) \geq -L_1\beta\hat{\mathbf{u}}'(\hat{z}_2). \tag{B.16}$$

Using (3.3a), we deduce that

$$\begin{aligned} 0 &= \delta\hat{\mathbf{u}}''(\hat{z}_2) - c\hat{\mathbf{u}}'(\hat{z}_2) + \mu(\Lambda - \hat{\mathbf{u}}(\hat{z}_2)) - (\beta\hat{\mathbf{u}}(\hat{z}_2) - \gamma)\hat{\mathbf{v}}(\hat{z}_2) \\ &\geq -\delta L_1\beta\hat{\mathbf{u}}'(\hat{z}_2) + cL_1(\beta\hat{\mathbf{u}}(\hat{z}_2) - \gamma) - (\beta\hat{\mathbf{u}}(\hat{z}_2) - \gamma)B \\ &\qquad \text{(by (B.15) and (B.16), and the fact that } \mu(\Lambda - \hat{\mathbf{u}}) > 0,\ \beta\hat{\mathbf{u}} - \gamma > 0 \text{ and } 0 < \hat{\mathbf{v}} \leq B) \\ &\geq \delta L_1^2\beta(\beta\hat{\mathbf{u}}(\hat{z}_2) - \gamma) + (\beta\hat{\mathbf{u}}(\hat{z}_2) - \gamma)B \quad \text{(by (B.15) and definition of } L_1) \\ &> 0, \quad \text{(use the fact that } \beta\hat{\mathbf{u}} - \gamma > 0) \end{aligned}$$

a contradiction again.

(2) We show that there exists a positive constant L_2 such that

$$\hat{\mathbf{u}}'(z) < L_2(\beta\hat{\mathbf{u}}(z) - \gamma), \forall z \geq 0. \tag{B.17}$$

Since $\hat{\mathbf{v}}$ is bounded, one can easily use (B.1) and (B.2) to deduce that $\mathcal{L}_2(\chi(\cdot))$ is bounded below on $[0, \infty)$. This, together with (3.8), implies that $\mathcal{L}_1(\chi(\cdot))$ is bounded above on $[0, \infty)$. Recall that

$$\mathcal{L}_1(\chi(z)) = -\left(\delta\hat{\mathbf{u}}'(z) - c\hat{\mathbf{u}}(z) - \delta(\mu + \kappa)\frac{\hat{\mathbf{u}}'(z)}{\beta\hat{\mathbf{u}}(z) - \gamma} + \frac{c(\mu + \kappa)}{\beta}\ln\frac{\beta\hat{\mathbf{u}}(z) - \gamma}{\beta s^* - \gamma}\right).$$

Then, using (B.3), the upper boundedness of $\mathcal{L}_1(\chi(\cdot))$, and the fact that $\gamma/\beta + \epsilon_0 \leq \hat{\mathbf{u}} < \Lambda$ on $\mathbb{R}$, we infer that $\hat{\mathbf{u}}'(z)/(\beta\hat{\mathbf{u}}(z) - \gamma)$ is bounded above over $z \geq 0$. Hence there exists a positive constant L_2 such that (B.17) holds. The proof of this lemma is thus completed. □

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科技部補助計畫衍生研發成果推廣資料表

日期:2016/10/14

科技部補助計畫	計畫名稱: SIS模型之旅行波解	
	計畫主持人: 符聖珍	
	計畫編號: 104-2115-M-004-002-	學門領域: 偏微分方程

無研發成果推廣資料

104年度專題研究計畫成果彙整表

計畫主持人：符聖珍				計畫編號：104-2115-M-004-002-		
計畫名稱：SIS模型之旅行波解						
成果項目				量化	單位	質化（說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）
國內	學術性論文	期刊論文		0	篇	
		研討會論文		0		
		專書		0	本	
		專書論文		0	章	
		技術報告		0	篇	
		其他		0	篇	
	智慧財產權及成果	專利權	發明專利 申請中	0	件	
			發明專利 已獲得	0		
			新型/設計專利	0		
		商標權		0		
		營業秘密		0		
		積體電路電路布局權		0		
		著作權		0		
		品種權		0		
		其他		0		
	技術移轉	件數		0	件	
		收入		0	千元	
國外	學術性論文	期刊論文		1	篇	submitted
		研討會論文		0		
		專書		0	本	
		專書論文		0	章	
		技術報告		0	篇	
		其他		0	篇	
	智慧財產權及成果	專利權	發明專利 申請中	0	件	
			發明專利 已獲得	0		
			新型/設計專利	0		
		商標權		0		
		營業秘密		0		
		積體電路電路布局權		0		
		著作權		0		
		品種權		0		
		其他		0		

	技術移轉	件數	0	件	
		收入	0	千元	
參與計畫人力	本國籍	大專生	0	人次	
		碩士生	1		
		博士生	0		
		博士後研究員	0		
		專任助理	0		
	非本國籍	大專生	0		
		碩士生	0		
		博士生	0		
		博士後研究員	0		
		專任助理	0		
其他成果 （無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）					

科技部補助專題研究計畫成果自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現（簡要敘述成果是否具有政策應用參考價值及具影響公共利益之重大發現）或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估
   ■達成目標
   □未達成目標（請說明，以100字為限）
      □實驗失敗
      □因故實驗中斷
      □其他原因
   說明：

2. 研究成果在學術期刊發表或申請專利等情形（請於其他欄註明專利及技轉之證號、合約、申請及洽談等詳細資訊）
   論文：□已發表 ■未發表之文稿 □撰寫中 □無
   專利：□已獲得 □申請中 ■無
   技轉：□已技轉 □洽談中 ■無
   其他：（以200字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性，以500字為限）
   The problem for existence of traveling wave solutions to the non-cooperative systems is challenging since there is no comparison principle and the phase space analysis for high-dimensional spaces is complicated. In this research, we provide a good way to deal with such problem. Our method might be applied to the study of waves in other non-cooperative systems (i.e., system without comparison principle).

4. 主要發現
   本研究具有政策應用參考價值：■否 □是，建議提供機關
   （勾選「是」者，請列舉建議可提供施政參考之業務主管機關）
   本研究具影響公共利益之重大發現：■否 □是
   說明：（以150字為限）