國立政治大學 108 學年度 碩士暨碩士在職專班 招生考試試題

考試科目	微積分	系所別	應用數學系	考試時間	2月17日(日)第一節

1. (12 %) Evaluate the limits.

   (a) (6 %) $\lim_{t\to 0}\frac{t^3}{\tan^2(2t)}$

   (b) (6 %) $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

2. (40 %) Evaluate the integrals.

   (a) (8 %) $\int_0^{\pi/4}(1+\tan t)^3\sec^2 t\,dt$

   (b) (8 %) $\int_0^{\pi/3}\sin x\ln(\cos x)dx$

   (c) (8 %) $\int_0^{\infty}e^{-\sqrt{y}}dy$

   (d) (8 %) $\int_0^{\infty}x^2e^{-x}dx$

   (e) (8 %) $\int_0^1\int_y^1 x^2e^{xy}dxdy$

3. (8 %) If $f(x)=\int_0^{g(x)}\frac{1}{\sqrt{1+t^3}}dt$, where $g(x)=\int_0^{\cos x}[1+\sin(t^2)]dt$, find $f'(\pi/2)$.

4. (10 %) Let

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^6} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

   Evaluate $f_{xy}(0,0)$.

5. (10 %) Evaluate the line integral

$$\int_C(yze^{xyz}+x)dx+xze^{xyz}dy+xye^{xyz}dz$$

   where C is the curve $\mathbf{r}(t)=\langle t,\cos(\pi t),\tan^{-1}(t)\rangle$, $0\le t\le 1$.

6. (10 %) Suppose f is a function with the property that $|f(x)|\le x^2$ for all x. Show that $f(0)=0$. Then show that $f'(0)=0$.

7. (10 %) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Prove that if $\lim_{n\to\infty}\frac{a_n}{b_n}=0$ and $\sum b_n$ converges, $\sum a_n$ also converges.

備註	一、作答於試題上者，不予計分 二、試題隨卷繳交

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國立政治大學 108 學年度 碩士班招生考試試題

考試科目	線性代數	系所別	應用數學系	考試時間	2 月 17 日 (日) 第二節

Please show all your work.

1. (10 points) Let V be the vector space of n-square matrices over a filed $\mathbb{R}$. Let U and W be the subspace of symmetric and antisymmetric matrices, respectively. Show that $V = U \oplus W$. (The matrix M is symmetric iff $M = M^t$, and antisymmetric iff $-M = M^t$.)

2. Let U and W be the subspaces of $\mathbb{R}^4$ generated by

$$\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\} \text{ and } \{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$$

respectively. Find each of the following:

   (a) (10 points) $dim(U+W)$

   (b) (10 points) $dim(U \cap W)$.

3. (10 points) Prove that a linear mapping $F: V \to U$ is nonsingular if and only if the image of a linearly independent set is linearly independent.

4. (10 points) Let $F : V \to U$ and $G : U \to W$ be linear. Show that $rank(G \circ F) \leq \min\{rank(F), rank(G)\}$

5. Let E be a linear operator on V for which $E^2 = E$. Let U be the image of E and W the kernel of E. Show that

   (a) (5 points) if $u \in U$, then $E(u) = u$

   (b) (5 points) if E is not the identity I on V, then E is singular.

   (c) (5 points) $V = U \oplus W$

6. (10 points) Let D be the differential operator on a vector space V of functions $f : \mathbb{R} \to \mathbb{R}$, that is, $D(f) = df/dt$. Find the matrix representation of D in the basis $\{e^{3t}, te^{3t}, t^2e^{3t}\}$ of V.

7. (10 points) Let $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where A and B are square matrices. Show that the minimum polynomial $m(t)$ of M is the least common multiple of the minimum polynomials $g(t)$ and $h(t)$ of A and B, respectively.

8. Let $T : V \to V$ be linear. Suppose, for $v \in V$, $T^k(v) = 0$ but $T^{k-1}(v) \neq 0$, that is, T is nilpotent of index k. Prove

   (a) (5 points) the set $S = \{v, T(v), \cdots, T^{k-1}(v)\}$ is linearly independent.

   (b) (5 points) the subspace W generated by S is T-invariant.

   (c) (5 points) the restriction $\hat{T}$ of T to W is nilpotent of index k.

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