Appendices

Proposition 5.2 *The sequence* $\{u_n(t)\}$ *defined by* (2.3), $\{u_n\} \subseteq \mathbb{X}_1$ *for all* $n \in \mathbb{N}$.

Proof. By mathematical induction

Step1. Let $u_0(t) = u_0$, so $u_0(t) \in C^1[0,T)$ and we can choose two positive numbers $M = \mid u_1 \mid +1$ and $N = \mid u_0 \mid +1$ such that

$$
\begin{aligned}
\mid u_0(t) \mid &= \mid u_0 \mid < N, \\
\mid u_0'(t) \mid &= 0 < M.
\end{aligned}
$$

Step2. $u_1(t) = u_0 + u_1 t$, so $u_1(t) \in C^1[0,T)$. For $t \in [0,T)$ since that $T < \dfrac{1}{\mid u_1 \mid}$,

$$
\begin{aligned}
\mid u_1(t) \mid &= \mid u_0 + u_1 t \mid \leq \mid u_0 \mid + \mid u_1 \mid t < N, \\
\mid u_1'(t) \mid &= \mid u_1 \mid < M.
\end{aligned}
$$

Step3. Suppose $u_{n-1}(t) \in C^1[0,T)$ and

$$
\begin{aligned}
&\mid u_{n-1}(t) \mid \leq N, \\
&\mid u_{n-1}'(t) \mid \leq M \quad \forall\, t \in [0,T).
\end{aligned}
$$

By Fundamental Theorem of Calculus, we have $u_n \in C^1[0,T)$ and

$$
\begin{aligned}
& \mid u_n(t) \mid \\
= & \mid u_0 + u_1 t + \int_0^t \int_0^r u_{n-1}'(s)^q (c_1 + c_2 u_{n-1}(s)^p)\; dsdr \mid \\
\leq & \mid u_0 \mid + \mid u_1 t \mid + \int_0^t \int_0^r \mid u_{n-1}'(s)^q (c_1 + c_2 u_{n-1}(s)^p) \mid \; dsdr \\
\leq & \mid u_0 \mid + \mid u_1 t \mid + \int_0^t \int_0^r \mid c_1 \mid M^q + \mid c_2 \mid M^q N^p \; dsdr \\
\leq & \mid u_0 \mid + \mid u_1 \mid T + (\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p) \frac{T^2}{2} \quad \text{for all } t \in [0,T)
\end{aligned}
$$

so

$$
\| u_n \|_\infty \leq \mid u_0 \mid + \mid u_1 \mid T + (\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p) \frac{T^2}{2}.
$$

Similarly

$$
\begin{aligned}
& \mid u_n'(t) \mid \\
= & \mid u_1 + \int_0^t u_{n-1}'(s)^q (c_1 + c_2 u_{n-1}(s)^p)\ ds \mid \\
\leq & \mid u_1 \mid + \int_0^t \mid u_{n-1}'(s)^q (c_1 + c_2 u_{n-1}(s)^p) \mid \ ds \\
\leq & \mid u_1 \mid + \int_0^t \mid c_1 \mid M^q + \mid c_2 \mid M^q N^p \ ds \\
\leq & \mid u_1 \mid + (\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p) T \quad \text{for all } t \in [0, T)
\end{aligned}
$$

so

$$\| u_n' \|_\infty \leq \mid u_1 \mid + (\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p) T.$$

Because

$$T \leq \ \min \left\{ \frac{1}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p}, \frac{- \mid u_1 \mid + \sqrt{u_1^2 + 2(\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p)}}{\mid c_1 \mid M^q + \mid c_2 \mid M^q N^p} \right\}.$$

we have $\| u_n \|_\infty \leq N$ and $\| u_n' \|_\infty \leq M$. ❑

Proposition 5.3 *If u and v are real value function, then*

$$|u^p(t) - v^p(t)| \leq p \ max\{ |u^{p-1}(t)|, |v^{p-1}(t)| \} \, |u(t) - v(t)| \quad \forall p \geq 1.$$

Proof. Let $f(k,t) = (ku(t) + (1-k)v(t))^p$, then $f(0,t) = v(t)^p$, $f(1,t) = u(t)^p$ and

$$|u^p(t) - v^p(t)| = |f(1,t) - f(0,t)| = | \int_0^1 f_1(r,t) dr|,$$

where $f_1 = \dfrac{\partial f(r,t)}{\partial r} = p(ru(t) + (1-r)v(t))^{p-1}(u(t) - v(t))$.
Therefore, for all $p \geq 1$, we conclude that

$$
\begin{aligned}
|u^p(t) - v^p(t)| &= | \int_0^1 p(ru(t) + (1-r)v(t))^{p-1}(u(t) - v(t)) dr| \\
&\leq p \int_0^1 |ru(t) + (1-r)v(t)|^{p-1} dr ||u(t) - v(t)| \\
&\leq p \int_0^1 (r|u(t)| + (1-r)|v(t)|)^{p-1} dr ||u(t) - v(t)| \\
&\leq p \max\{ |u^{p-1}(t)|, |v^{p-1}(t)| \} \, |u(t) - v(t)| \quad \forall p \geq 1.
\end{aligned}
$$
❑