6 Number of Unbounded Regions

In this section, we discuss the number of unbounded regions in the same condition as bounded regions. We can easily obtain the formula of such number by subtracting the number of bounded regions since we already have the number of total regions and the number of bounded regions. We state all the questions in the following:

Number of Unbounded Regions in Sense of k-max-point-drawing How many unbounded regions are there in sense of a k-max-point-drawing?

Number of Unbounded Regions in Sense of k-max-line-drawing How many unbounded regions are there in sense of a k-max-line-drawing?

Number of Unbounded Regions in Sense of k-max-plane-drawing How many unbounded regions are there in sense of a k-max-plane-drawing?

6.1 Number of Unbounded Regions in Sense of k-max-point-drawing

How many unbounded regions are there in sense of k-max-point-drawing?

Let ${a_k}^{**}$ be the number of unbounded regions in sense of k-max-point drawing.

The number of total regions is $C_0^k + C_1^k$ and the number of bounded regions is C_1^{k-1}. Hence, we have the formula of ${a_k}^{**}$ in the following:

$$\begin{aligned}
{a_k}^{**} &= a_k - {a_k}^{*} \\
&= (C_0^k + C_1^k) - C_1^{k-1} \\
&= C_0^k + (C_0^{k-1} + C_1^{k-1}) - C_1^{k-1} \\
&= C_0^k + C_0^{k-1} \\
&= 2C_0^k
\end{aligned}$$

6.2 Number of Unbounded Regions in Sense of k-max-line-drawing

How many unbounded regions are there in sense of k-max-line-drawing?

Let ${b_k}^{**}$ be the number of unbounded regions in sense of k-max-line drawing.

Again, we use the same way to get the formula ${b_k}^{**}$ in the following:

$$\begin{aligned}
{b_k}^{**} &= b_k - {b_k}^{*} \\
&= (C_0^k + C_1^k + C_2^k) - C_2^{k-1} \\
&= C_0^{k-1} + (C_0^{k-1} + C_1^{k-1}) + (C_1^{k-1} + C_2^{k-1}) - C_2^{k-1} \\
&= C_0^{k-1} + C_0^{k-1} + C_1^{k-1} + C_1^{k-1} \\
&= C_1^k + C_1^k \\
&= 2C_1^k
\end{aligned}$$

6.3 Number of Unbounded Regions in Sense of k-max-plane-drawing

How many unbounded regions are there in sense of k-max-plane-drawing?

Let ${c_k}^{**}$ be the number of unbounded regions in sense of k-max-plane drawing.

Once again, using the same way to get the formula ${c_k}^{**}$ in the following:

$$\begin{aligned}
{c_k}^{**} &= c_k - {c_k}^{*} \\
&= (C_0^k + C_1^k + C_2^k + C_3^k) - C_3^{k-1} \\
&= C_0^k + (C_0^{k-1} + C_1^{k-1}) + (C_1^{k-1} + C_2^{k-1}) + (C_2^{k-1} + C_3^{k-1}) - C_3^{k-1} \\
&= C_0^k + C_0^{k-1} + C_1^{k-1} + C_1^{k-1} + C_2^{k-1} + C_2^{k-1} \\
&= (C_0^k + C_0^{k-1}) + (C_1^{k-1} + C_1^{k-1} + C_2^{k-1} + C_2^{k-1}) \\
&= 2C_0^k + 2C_2^k
\end{aligned}$$

6.4 Number of Unbounded Regions of Higher Dimensional Space

Now, we generalize the questions into the following one:

How many unbounded regions are there in an n-dimensional space if k different $(n-1)$-dimensional spaces partition an n-dimensional space into the largest number of different parts?

Let ${P_{n,k}}^{**}$ be such a number of unbounded regions in this question. Here, n means the n-dimensional space and k means a set of k partitioner in this space (i.e. k different $(n-1)$-dimensional spaces).

All the same, we don't have a clear definition about the unbounded region in an n-dimensional space made by a set of $(n-1)$-dimensional spaces, but we still can obtain the formula of ${P_{n,k}}^{**}$ by subtracting ${P_{n,k}}^{*}$, the number of bounded regions, since we have a conjecture about it. But the difference is that the formula about unbounded regions is separated into two parts by its parity of dimension.

For the odd dimensional space (i.e. n is odd), we have the following formula:

$$\begin{aligned}
{P_{n,k}}^{**} &= P_{n,k} - {P_{n,k}}^{*} \\
&= (C_0^k + C_1^k + \cdots + C_n^k) - C_n^{k-1} \\
&= C_0^k + (C_0^{k-1} + C_1^{k-1}) + (C_1^{k-1} + C_2^{k-1}) + \cdots + (C_{n-1}^{k-1} + C_n^{k-1}) - C_n^{k-1} \\
&= C_0^k + C_0^{k-1} + C_1^{k-1} + C_1^{k-1} + C_2^{k-1} + C_2^{k-1} + \cdots + C_{n-1}^{k-1} + C_{n-1}^{k-1} \\
&= (C_0^k + C_0^{k-1}) + (C_1^{k-1} + C_1^{k-1} + C_2^{k-1} + C_2^{k-1}) + \cdots \\
&\quad + (C_{n-2}^{k-1} + C_{n-2}^{k-1} + C_{n-1}^{k-1} + C_{n-1}^{k-1}) \\
&= 2C_0^k + 2C_2^k + \cdots + 2C_{n-1}^k
\end{aligned}$$

It is because that n is odd and each combination C is equal to the sum of two combination C by the equality $C_n^m = C_{n-1}^{m-1} + C_n^{m-1}$, there are $2n$ Cs (i.e. a multiple of 2) in the fourth equality. Therefore, as we use parentheses for every 4 Cs in the backward order, there still has 2 Cs in the fifth equality.

For the even dimensional space (i.e. n is even), we have the following formula:

$$
\begin{aligned}
{P_{n,k}}^{**} &= P_{n,k} - {P_{n,k}}^{*} \\
&= (C_0^k + C_1^k + \cdots + C_n^k) - C_n^{k-1} \\
&= C_0^{k-1} + (C_0^{k-1} + C_1^{k-1}) + (C_1^{k-1} + C_2^{k-1}) + \cdots + (C_{n-1}^{k-1} + C_n^{k-1}) - C_n^{k-1} \\
&= C_0^{k-1} + C_0^{k-1} + C_1^{k-1} + C_1^{k-1} + C_2^{k-1} + C_2^{k-1} + \cdots + C_{n-1}^{k-1} + C_{n-1}^{k-1} \\
&= (C_0^{k-1} + C_0^{k-1} + C_1^{k-1} + C_1^{k-1}) + \cdots + (C_{n-2}^{k-1} + C_{n-2}^{k-1} + C_{n-1}^{k-1} + C_{n-1}^{k-1}) \\
&= 2C_1^k + 2C_3^k + \cdots + 2C_{n-1}^k
\end{aligned}
$$

Here, n is even and each combination C is still equal to the sum of two combination C by the same equality $C_n^m = C_{n-1}^{m-1} + C_n^{m-1}$, there are $2n$ Cs (i.e. a multiple of 4) in the fourth equality. Hence, we use up all combination Cs as we make parentheses for every 4 Cs in the backward order.