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非線性微分方程的週期解
PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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中文摘要

我們討論具週期邊界條件之邊界值問題. 當上解大於下解或下解大於上解時, 我們針對二階積分微分方程進行分析, 並利用混合單調法探討週期解之存在性.

關鍵詞：上(下)解, 積分微分方程, 混合單調法.

Periodic Solutions of Nonlinear Differential Equations*

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Abstract

We consider second order integro-differential systems with periodic boundary conditions. Assuming that the upper solution is less than the lower solution, the existence of periodic solution by the method of mixed monotony is obtained.

1 Introduction.

In this paper we shall consider periodic boundary value problem (PBVP) of second order integro-differential system of the form

$$-u_i'' = f_i\left(t, u, T^i u\right), \quad 0 < t < 2\pi, \quad i = 1, 2, ..., N, \tag{1}$$

with boundary condition

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \tag{2}$$

where $f \in C\left(I \times R^N \times R^N, R^N\right)$, $I = [0, 2\pi]$, and $T^i = \left(T^{i1}, ..., T^{iN}\right)$, $1 \leq i \leq N$, are some bounded integral operators which are nondecraesing. For example,

$$\left(T^{ij}u\right)(t) = \int_0^t a^{ij}(s)\, u_j(s)\, ds, \quad t \in I$$

where $a^{ij}(s)$ are nonnegative functions on I.

There are many results for the periodic boundary value problem without integral terms, see [2] and references therein., The method of upper and lower solutions is widely used to discuss the existence, uniqueness, boundedness, stability and asymptotic behavior of the solutions. Usually it is assumed that the upper solution is larger than the lower solution when we use monotone method. The question is whether the existence of solution holds by reversing the order of the upper and lower solutions. Some results are given in [2]. On the other

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hand, the method of mixed monotony has been used to discuss the existence of boundary value problems by [1], [3] and [4] under the assumption that the upper solution is greater than the lower solution. In this note we shall consider the reverse order case and discuss the existence of the solution of periodic BVP by the method of mixed monotony. This will complement the result of [1] and generalize the result in section 5.3 of [2]. For convenience, we give the notation :

$$[\beta,\alpha] = \left\{u \in C\left(I, R^N\right) | \beta \le u \le \alpha \text{ on } I\right\}$$

here $\beta \le u$ means that $\beta_i \le u_i$ for $1 \le i \le N$.

2 Preliminaries.

THEOREM 1 ([1]): Assume that there exist a positive constant ε, a function $F \in C\left(I \times R^N \times R^N \times R^N \times R^N, R^N\right)$, and two functions $\alpha, \beta \in C^2\left(I, R^N\right)$ such that the following conditions hold :

(A1) $\alpha(t) \le \beta(t)$, $t \in I$.

(A2) for all $u, v \in C\left(I, R^N\right)$ with $\alpha \le u \le \beta$ and $\alpha \le v \le \beta$, we have

$$\begin{gathered}\alpha_i'' + F_i\left(t, u, T^i u, v, T^i v\right) \ge -\tfrac{1}{2\varepsilon}\left[(v_i - \alpha_i) + (u_i - \alpha_i)\right] \quad \text{on } I, \\ \alpha(0) = \alpha(2\pi), \quad \alpha'(0) \ge \alpha'(2\pi).\end{gathered}$$

(A3) for all $u, v \in C\left(I, R^N\right)$ with $\alpha \le u \le \beta$ and $\alpha \le v \le \beta$, we have

$$\begin{gathered}\beta_i'' + F_i\left(t, u, T^i u, v, T^i v\right) \le -\tfrac{1}{2\varepsilon}\left[(v_i - \beta_i) + (u_i - \beta_i)\right] \quad \text{on } I, \\ \beta(0) = \beta(2\pi), \quad \beta'(0) \le \beta'(2\pi).\end{gathered}$$

(A4) $F_i(t, u, y, v, z)$ is nondecreasing in u and y and nonincreasing in v and z respectively for fixed the remaining arguments.

(A5) $F_i\left(t, u, T^i u, u, T^i u\right) = f_i\left(t, u, T^i u\right)$, $1 \le i \le N$.

(A6) If there exist two functions $\rho, \gamma \in C^2\left(I, R^N\right)$ such that

$$\begin{gathered}\rho_i'' + F_i\left(t, \rho, T^i \rho, \gamma, T^i \gamma\right) = -\tfrac{1}{2\varepsilon}\left(\gamma_i - \rho_i\right) \quad \text{on } I, \\ \rho(0) = \rho(2\pi), \quad \rho'(0) = \rho'(2\pi),\end{gathered}$$

and

$$\begin{gathered}\gamma_i'' + F_i\left(t, \gamma, T^i \gamma, \rho, T^i \rho\right) = -\tfrac{1}{2\varepsilon}\left(\rho_i - \gamma_i\right) \quad \text{on } I, \\ \gamma(0) = \gamma(2\pi), \quad \gamma'(0) = \gamma'(2\pi),\end{gathered}$$

then $\rho \equiv \gamma$ on I.

Then the problem $(1), (2)$ has a unique solution u with $\alpha(t) \le u(t) \le \beta(t)$ on I.

LEMMA 2 ([2]): Assume that $g \in C(I)$, $g \ge 0$ and g is not trivial. Then the boundary value problem

$$\begin{gathered}u'' + ku = g(t) \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi) + \lambda\end{gathered}$$

has a positive solution on I for any $\lambda \ge 0$ if and only if $0 < k \le \frac{1}{4}$.

3 Main results.

We shall consider the existence of the solution for periodic boundary value problem (1),(2) under the assumption that the upper solution is less than the lower solution. The method of mixed monotony will be used.

THEOREM 3 : Assume that there exist two functions $\alpha, \beta \in C^2(I, R^N)$ and a function $F \in C(I \times R^N \times R^N \times R^N \times R^N, R^N)$ such that g the following conditions hold :
(B1) $\beta(t) \leq \alpha(t)$ on I,
(B2) for all $u, v \in C(I, R^N)$ with $\beta \leq u \leq \alpha$ and $\beta \leq v \leq \alpha$, we have

$$\alpha_i'' + F_i(t, u, T^i u, v, T^i v) \geq \tfrac{1}{8}[(v_i - \alpha_i) + (u_i - \alpha_i)] \quad \text{on } I,$$
$$\alpha(0) = \alpha(2\pi), \quad \alpha'(0) \geq \alpha'(2\pi).$$

(B3) for all $u, v \in C(I, R^N)$ with $\beta \leq u \leq \alpha$ and $\beta \leq v \leq \alpha$, we have

$$\beta_i'' + F_i(t, u, T^i u, v, T^i v) \leq \tfrac{1}{8}[(v_i - \beta_i) + (u_i - \beta_i)] \quad \text{on } I,$$
$$\beta(0) = \beta(2\pi), \quad \beta'(0) \leq \beta'(2\pi).$$

(B4) $F_i(t, u, y, v, z)$ is nonincreasing in u and y and nondecreasing in v and z respectively.
(B5) $F_i(t, u, T^i u, u, T^i u) = f_i(t, u, T^i u)$, $1 \leq i \leq N$
(B6) If there exist two functions $\rho, \gamma \in C^2(I, R^N)$ such that

$$\rho_i'' + F_i(t, \rho, T^i \rho, \gamma, T^i \gamma) = \tfrac{1}{8}(\gamma_i - \rho_i),$$
$$\rho(0) = \rho(2\pi), \quad \rho'(0) = \rho'(2\pi),$$

and

$$\gamma_i'' + F_i(t, \gamma, T^i \gamma, \rho, T^i \rho) = \tfrac{1}{8}(\rho_i - \gamma_i),$$
$$\gamma(0) = \gamma(2\pi), \quad \gamma'(0) = \gamma'(2\pi),$$

then $\rho \equiv \gamma$ on I.
Then the problem (1),(2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

Some sufficient conditions for the existence of F in theorem 3 are given below.

THEOREM 4 : Assume that there exist two functions $\alpha, \beta \in C^2(I, R^N)$ satisfying the following conditions :
(C1) $\beta(t) \leq \alpha(t)$ on I,
(C2) for all $u \in C(I, R^N)$ with $\beta \leq u \leq \alpha$ on I, we have

$$\alpha_i'' + f_i(t, u, T^i u) \geq \tfrac{1}{8}(u_i - \alpha_i) \quad \text{on } I,$$
$$\alpha(0) = \alpha(2\pi), \quad \alpha'(0) \geq \alpha'(2\pi).$$

(C3) for all $u \in C(I, R^N)$ with $\beta \leq u \leq \alpha$, we have

$$\begin{gathered} \beta_i'' + f_i(t, u, T^i u) \leq \tfrac{1}{8}(u_i - \beta_i) \text{ on } I, \\ \beta(0) = \beta(2\pi), \quad \beta'(0) \leq \beta'(2\pi). \end{gathered}$$

(C4) $f_i(t, u, y)$ is nonincreasing in u and y for each fixed $t \in I$.
Then the boundary value problem (1), (2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

Analogously, the variant of theorem 3 can be obtained as follows :

THEOREM 5 :Assume that there exist two functions $\alpha, \beta \in C^2(I, R^N)$ and a function $F \in C(I \times R^N \times R^N \times R^N \times R^N, R^N)$ such that the following conditions hold :
(D1) $\beta(t) \leq \alpha(t)$ on I.
(D2)

$$\begin{gathered} \alpha_i'' + F_i(t, \alpha, T^i\alpha, \beta, T^i\beta) \geq 0 \text{ on } I, \\ \alpha(0) = \alpha(2\pi), \quad \alpha'(0) \geq \alpha'(2\pi). \end{gathered}$$

(D3)

$$\begin{gathered} \beta_i'' + F_i(t, \beta, T^i\beta, \alpha, T^i\alpha) \leq 0 \text{ on } I, \\ \beta(0) = \beta(2\pi), \quad \beta'(0) \leq \beta'(2\pi). \end{gathered}$$

(D4) For fixed t, u, y, we have

$$F_i(t, u, y, v, z) \geq F_i(t, u, y, v^*, z^*) \text{ for } v \geq v^*, z \geq z^*,$$

and for fixed t, v, z,

$$F_i(t, u^*, y^*, v, z) - F_i(t, u, y, v, z) - \frac{1}{4}(u_i^* - u_i) \leq 0 \text{ for } u^* \geq u, y^* \geq y.$$

(D5) $F_i(t, u, T^i u, u, T^i u) = f_i(t, u, T^i u)$, $1 \leq i \leq N$.
(D6) If there exist two functions $\rho, \gamma \in C^2(I, R^N)$ such that

$$\begin{gathered} \rho_i'' + F_i(t, \rho, T^i\rho, \gamma, T^i\gamma) = 0 \text{ on } I, \\ \rho(0) = \rho(2\pi), \quad \rho'(0) = \rho'(2\pi), \end{gathered}$$

and

$$\begin{gathered} \gamma_i'' + F_i(t, \gamma, T^i\gamma, \rho, T^i\rho) = 0 \text{ on } I, \\ \gamma(0) = \gamma(2\pi), \quad \gamma'(0) = \gamma'(2\pi), \end{gathered}$$

then $\rho \equiv \gamma$ on I.
Then the problem (1), (2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

THEOREM 6 : Assume that there exist two functions $\alpha, \beta \in C^2(I, R^N)$ satisfying the following conditions :

(E1) $\beta(t) \leq \alpha(t)$ on I.

(E2)

$$\alpha_i'' + f_i(t, \alpha, T^i\alpha) \geq 0 \text{ on } I,$$
$$\alpha(0) = \alpha(2\pi), \quad \alpha'(0) \geq \alpha'(2\pi).$$

(E3)

$$\beta_i'' + f_i(t, \beta, T^i\beta) \leq 0 \text{ on } I,$$
$$\beta(0) = \beta(2\pi), \quad \beta'(0) \leq \beta'(2\pi).$$

(E4) For $u^* \geq u, y^* \geq y$, we have

$$f_i(t, u^*, y^*) - f_i(t, u, y) - \frac{1}{4}(u_i^* - u_i) \leq 0.$$

Then the boundary value problem (1), (2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

4 References.

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