行政院國家科學委員會補助專題研究計畫成果報告

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計畫主持人：陳天進 教授

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行政院國家科學委員會專題研究計畫成果報告

複解析動態系統
Holomorphic Mappings on $\mathbb{C}^n$

計畫編號：NSC 89-2115-M-004-003-
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一、中文摘要

本計劃乃延續去年的計劃，繼續探討複空間之解析映射及動態系統，其主要探討的領域描繪如下：

(I). 探討複空間的一些特殊自同構，例如，Shears 和下三角映射。
(II). 探討 Fatou-Bierberbach 域之幾何性及邊界行為。
(III). 探討 Fatou-Bierberbach 域與雙曲測度之關係。

關鍵詞：自同構 Fatou-Bierberbach 域、複 Henon 寫像、吸引盤、Shear、下三角映像、雙曲測度。

Abstract

In this project, we will continue the last project to study holomorphic mappings from C^n into C^n $(n \geq 2)$ and their dynamics system. The most interesting research area are described as follows:

(I) Some special automorphisms of C^n, for example, shears and lower triangular mappings.
(II) The geometry of a Fatou-Bierberbach domain and its boundary behavior.
(III) The relation of Fatou-Bierberbach domain and measure hyperbolicity.

Keywords: Automorphism of C^n, Complex Henon map, Basin of attraction, Fatou-Bierberbach domain, Measure hyperbolic

二、內容

ASBTRACT. In this report, we will collect some well-known Automorphisms of $\mathbb{C}^n$ and investigate the complex Henon maps and the Fatou-Bieberbach domains in $\mathbb{C}^n$.

§1 Automorphisms of $\mathbb{C}^n$

Let $\text{Aut}(\mathbb{C}^n)$ denote the automorphism group of $\mathbb{C}^n$, i.e.,

$$\text{Aut}(\mathbb{C}^n) = \left\{ F : \mathbb{C}^n \to \mathbb{C}^n \middle| \begin{array}{l} \text{F is a biholo} \\ \text{morphic map} \end{array} \right\}.$$

With the composition of map, it is a group. For $n=1$, it is well known in complex analysis that $\text{Aut}(\mathbb{C}^n)$ contains exactly the class of all affine linear maps of $\mathbb{C}$, i.e.,

$$\text{Aut}(\mathbb{C}) = \left\{ f : \mathbb{C} \to \mathbb{C} \middle| \begin{array}{l} f(z) = az + b \\ \text{for some } a, b \in \mathbb{C}, a \neq 0 \end{array} \right\}$$

For $n \geq 2$, $\text{Aut}(\mathbb{C}^n)$ is very large and complicated. In [3,5], the authors listed some automorphisms of $\mathbb{C}^n$, we collect them as follows:

(i) $F_t(z) = z + t\,f(T(z))u,\ z \in \mathbb{C}^n$, where $t \in \mathbb{C}$, $u = \mathbb{C}^n$ and $T:\mathbb{C}^n \to \mathbb{C}^k$ is a complex linear map, $1 \le k < n$, $f:\mathbb{C}^k \to \mathbb{C}^k$ is entire and $T(u) = 0$. Such F_t is called a shear in the direction u which was introduced by Rosay and Rudin.

(ii) $G_t(z) = z + \left(e^{tg(T(z))} - 1\right)\langle z, u\rangle u,\ z \in \mathbb{C}^n$, where $t \in \mathbb{C}$, $u \in \mathbb{C}^n$ and $T:\mathbb{C}^n \to \mathbb{C}^k$ is a complex linear map, $1 \le k < n$, and $T(u) = 0$, $g:\mathbb{C}^k \to \mathbb{C}^k$, $\langle z, u\rangle = \sum_{j=1}^{n} z_j \bar{u}_j$. Such G_t is called a generalized shear in the direction u.

(iii) $S_t(z) = z + t\,h(\omega(z,u))u,\ z \in \mathbb{C}^{2n}$ where $t \in \mathbb{C}$, $u \in \mathbb{C}^{2n}$, $h:\mathbb{C} \to \mathbb{C}$ is entire and $\omega = \sum_{j=1}^{n} dz_j \wedge dz_{n+j}$ is the sympletic form on $\mathbb{C}^{2n}$.

(iv) $w = F(z)$, where

$F(z) = (F_1(z), F_2(z), \cdots, F_n(z))$ and

$F_j(z) = z_j \exp\left(c_j f(z_1^{a_1} \cdots z_n^{a_n})\right)$, $1 \le j \le n$, $z \in \mathbb{C}^n$, $a_1, a_2, \cdots, a_n$, are nonnegative integers, $c_j \in \mathbb{C}$, $1 \le j \le n$, $\sum_{j=1}^{n} c_j a_j = 0$ and $f:\mathbb{C} \to \mathbb{C}$ is entire.

(v) $G = (g_1, g_2, \cdots, g_n):\mathbb{C}^n \to \mathbb{C}^n$, where

$$g_1(z) = c_1 z_1$$
$$g_2(z) = c_2 z_2 + h_2(z_1)$$
$$\vdots$$
$$g_n(z) = c_n z_n + h_n(z_1, \cdots, z_{n-1})$$

, $c_1, c_2, \cdots, c_n \in \mathbb{C}$ are nonzero and $h_k:\mathbb{C}^{k-1} \to \mathbb{C}$ is entire for $2 \le k \le n$. Such automorphism is called a lower triangular map, which was introduced by Rosay and Rudin.

All these automorphisms are used in [3,5] to study the interpolation and density problems. One can find the details there.

§ 2 Complex Henon Maps and Fatou-Bieberbach Domains

Definition.

A Fatou-Bieberbach domain in $\mathbb{C}^n$, $n \ge 2$, is a proper subdomain in $\mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$.

Definition.

Let $F \in \mathrm{Aut}(\mathbb{C}^n)$, $p \in \mathbb{C}^n$ with $F(p) = p$ and F^j are the *j*-fold composition $F \circ F \circ \cdots \circ F$ of F. The basin of attraction of F is define to be

$$\Omega = \left\{ z \in \mathbb{C} \,\middle|\, \lim_{j \to \infty} F^j(z) = p \right\}$$

If Ω is a Fatou-Bieberbach domain in $\mathbb{C}^n$, then F is called a complex Henon map.

In [1,2,4,5], the author constructed a collection of Fatou-Bieberbach domains and complex Henon maps in $\mathbb{C}^n$, we list some of them as follows:

Theorem 1 [4]

Given $a \in \mathbb{C}$, $0 < |a| < 1$. The map

$$F(z) = \left(z_1^2 + a z_2, a z_1\right),\ z \in \mathbb{C}^2$$

is a complex Henon map in $\mathbb{C}^2$ whose basin of attraction at the origin is a Fatou-Bieberbach domain in $\mathbb{C}^2$.

Theorem 2 [2]

Let $a \in \mathbb{C}$, $0 < |a| < 1$. The map

$$F(z) = \left(z_1^2 + a z_2, z_2^2 + a z_3, a z_1\right),\ z \in \mathbb{C}^3$$

is a complex Henon map in $\mathbb{C}^3$ whose basin of attraction at the origin is a Fatou-Bieberbach domain in $\mathbb{C}^3$.

Combine Theorem 1 and 2, we obtain a collection of complex Henon maps and

Fatou-Bieberbach domains in $\mathbb{C}^n$, $n \geq 4$.

Corollary

(a) if n is even, say $n = 2k$, then $F = (F_1, F_2, \cdots, F_k)$, where $F_1, F_2, \cdots, F_k$ are complex Henon maps defined in Theorem 1 is a complex Henon map in $\mathbb{C}^n$ whose basin of attraction at the origin is a Fatou-Bieberbach domain in $\mathbb{C}^n$.

(b) if n is odd, say $n = 2k+1$, then $F = (F_1, F_2, \cdots, F_k)$, where $F_1, F_2, \cdots, F_{k-1}$ are complex Henon maps defined in Theorem 1 and F_k is a complex Henon map defined in Theorem 2, is a complex Henon map in $\mathbb{C}^n$ whose basin of attraction at the origin is a Fatou-Bieberbach domain in $\mathbb{C}^n$.

Theorem 3 [5]

Suppose that $F \in \mathrm{Aut}(\mathbb{C}^n)$, $F(0) = 0$ and all eigenvalues λ_i of $DF(0)$ satisfy $|\lambda_i| < 1$, $1 \leq k \leq n$, where $DF(0)$ is the complex Jacobian matrix of F at 0. Let Ω be the basin of attraction of F at 0, i.e.,

$$\Omega = \left\{ z \in \mathbb{C}^n \,\middle|\, \lim_{j \to \infty} F^j(z) = 0 \right\}$$

Then Ω is biholomorphic to $\mathbb{C}^n$.

Theorem 3 says only that Ω is biholomorphic to $\mathbb{C}^n$, therefore, F may not be a complex Henon map in $\mathbb{C}^n$, for example, $F(z) = (\lambda_1 z_1, \lambda_2 z_2, \cdots, \lambda_n z_n)$, where $0 < |\lambda_i| < 1$, $1 \leq j \leq n$, is a complex linear map on $\mathbb{C}^n$, in this case, the basis of attraction Ω of F at the origin is $\mathbb{C}^n$ itself, so F is not a complex Henon map in particular Ω is not a Fatou-Bieberbach domain in $\mathbb{C}^n$.

Now, we consider the class of lower triangular maps defined in (V), **§** 1.

Let $G(z) = (g_1(z), g_2(z), \cdots, g_n(z))$ be a lower triangular map defined by

$$g_1(z) = \lambda_1 z_1$$
$$g_2(z) = \lambda_2 z_2 + h_2(z_1)$$
$$\vdots$$
$$g_n(z) = \lambda_n z_n + h_n(z_1, \cdots, z_{n-1})$$

where $\lambda_j \in \mathbb{C}$, $0 < |\lambda_i| < 1$ for all $1 \leq j \leq n$ and $h_j \in \mathbb{C}^{j-1} \to \mathbb{C}$ is entire, $h_j(0) = 0$, $2 \leq j \leq n$.

Clearly, $G(0) = 0$, i.e., 0 is a fixed point of G, and the eigenvalues of $DG(0)$ are $\lambda_1, \lambda_2, \cdots \lambda_n$ which satisfy $0 < |\lambda_i| < 1$ for all $1 \leq j \leq n$. Therefore, by Theorem 3, the basin of attraction of F at the origin

$$\Omega = \left\{ z \in \mathbb{C}^n \,\middle|\, \lim_{j \to \infty} G^j(z) = 0 \right\}$$

is biholomorphic to $\mathbb{C}^n$. Therefore, the following question arises naturally: when Ω is a Fatou-Bieberbach domain? Equivalently, can one put some sufficient conditions on $h_2, h_3, \cdots h_n$ so that G be comes a complex Henon map in $\mathbb{C}^n$? We will continue the study of this problem in the Fatou-Bieberbach domains are: the geometry of a Fatou-Bieberbach domain Ω in $\mathbb{C}^n$, especially, for $\partial\Omega$, the measure hyperbolicity of a such domain and how many complex lines can a Fatou-Bieberbach domain in $\mathbb{C}^n$ contain?

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