行政院國家科學委員會補助專題研究計畫成果報告

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計畫主持人： 陸 行
共同主持人：

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行政院國家科學委員會專題研究計畫成果報告
佇列中離去時間的相關性質
The Correlation Structure of the Output Process in A Queue

計畫編號 : **NSC 89-2118-M-004-002**
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1 Abstract

In this report, we consider a general $PH/G/1$ queueing system in which the interarrival time follows a phase-type renewal process and service time distribution is general. Correlation of consecutive interdeparture times characterizing the output process will be studied. We shall first construct a recursive procedure for calculating the joint probability distribution of an arbitrary number of consecutive indeparture times in a $PH/G/1$ queue. Closed form solutions of the equilibrium distribution are derived for this model and the Laplace-Stieltjes transform (LST) of the distribution of interdeparture times is presented. We then obtain explicitly the covariances of nonadjacent interdeparture times, and display the correlation coefficients that reveal the long-range dependence.

(Keywords : $PH/G/1$ Queueing, Matrix-geometric solutions, Covariances of Departure processes)

遞迴過程系統，再導出這個模型封閉型式的解和離去時間的拉氏轉換 *(LST)*。我們因此可得到不相連的離去時間之相關變異數和顯示大範圍的相關係數

(關鍵字 :$PH/G/1$ 等候系統，幾何矩陣的解，進入過程的相關變數)

2 Introduction

[illegible] of $GI/G/1$ [illegible] of many researchers. [illegible] closed form solutions for the distribution of the departure process are nearly intractable. Burke [2] and Daley [6] investigated departure processes from a $GI/M/1$ queue and studied the correlation structure. He proved that the output process of a stationary $GI/M/1$ queueing system is a renewal process if and only if the input process is a Poisson process, in which case the output process is a Poisson process. Chang [3] discussed a general phenomenon in queueing theory. He showed that the Poisson process is the only stationary and ergodic process that induces identical distributions on the interdeparture times when the service times are exponentially distributed. For a departure process in a specific model, Takagi and Nishi [15] employed probability decomposition and Laplace-Stieltjes transforms to derive correlation coefficients of interdeparture times for $M/G/1$ and $M/G/1/K$ queues.

Jenkins [11] analyzed the correlation of consecutive interdeparture times for an $M/E_m/1$ queue, where E_m denotes an Erlang distribution with m phases. For a $M/G/1$ queue, Conolly [4] gave the joint distribution [illegible] and Ishikawa [10] have studied the correlation structure of the $M/G/1/K$ queue. King [12] also had an investigation on the covariance structure of the output process. But their results were only limited on the case where the arrival process is Poisson. In the report, we will construct a procedure for calculating the joint probability distribution for the arbitrary number n consecutive interdeparture times $D_1, D_2, \ldots, D_n$ for a general arrival process. Our method provides a different approach by taking advantage of a recursive structure in the set of interdeparture times. The methodology is based on the development of Luh [13].

The departure process

$\mathbf{f}_n^*(s_n, s_{n-1}, \ldots, s_1)$ be the Laplace transform of the joint probability density function for n consecutive interdeparture times D_1 through D_n where the transform parameter s_i corresponds to D_{n+1-i}. Let $\mathbf{f_n^*(s_n, s_{n-1}, \ldots, s_1 \mid k)}$ be the LST of the joint probability density function for n consecutive interdeparture times τ_0 through τ_n condition on k customers in the queue at a departure time. It is an M dimensional column vector whose elements are associated with each phase of the arrival process. Let $\mathbf{E}_i(y)$ be a probability conditioned on y.
Obviously,

$$\mathbf{f}_1^*(s_1 \mid 1) = (H^*(s_1), H^*(s_1), \ldots, H^*(s_1))^T$$

$$\mathbf{f}_1(s_1 \mid 0) = H^*(s_1)\mathbf{a}^*(s_1)$$

$$\mathbf{f}_n^*(s_n, s_{n-1}, \ldots, s_1 \mid k) = \mathbf{u}^*(s_n, s_{n-1}, \ldots, s_1)\cdot$$

$$\mathbf{d}^*(s_n \mid n-k) + \sum_{i=0}^{n-k-1} \mathbf{Q}_i^*(s_n)\cdot$$

$$f_{n-1}^*(s_n, s_{n-1}, \ldots, s_1 \mid n-k-1)$$

where $\mathbf{c}^*(s) = (a_1(s), a_2(s), \ldots, a_M(s))^T$

$$\mathbf{Q}_i^* = \int_{y \geq 0} e^{-sy}\mathbf{E}_i(y)dH(y), 1 \leq k \leq n-1$$

$$u^*(s_{n-1}, s_{n-2}, \ldots, s_1) = \prod_{i=1}^{n-1} H^*(s_i)$$

$$\mathbf{d}^*(s \mid m) = [d_1^*(s \mid m), d_2^*(s \mid m), \ldots, d_M^*(s \mid m)]^T$$

$$\mathbf{d}_j^*(s \mid m) = \int_{y \geq 0} e^{-sy}\beta_{j,m,M}(y)dH(y)$$

$$\mathbf{f}_n^*(s_n, s_{n-1}, \ldots, s_1 \mid n) =$$

$$[u^*(s_n, s_{n-1}, \ldots, s_1), u^*(s_n, s_{n-1}, \ldots, s_1), \ldots,$$

$$u^*(s_n, s_{n-1}, \ldots, s_1]^T$$

$$\mathbf{f}_n^*(s_n, s_{n-1}, \ldots, s_1 \mid 0) =$$

$$f_n^*(s_n, s_{n-1}, \ldots, s_1 \mid 1)^T \cdot \mathbf{e}_k a^*(s_n)^T$$

We will now consider the joint p.d.f. of D_{n-1} and D_n. Using conditional probabilities this joint p.d.f. is equal to $\mathbf{f}_n^*(s_n, s_{n-1}, \ldots, s_1 \mid k)$. The marginal joint p.d.f. of D_{n-1} and D_n is given by $\mathbf{f}_n^*(s_n, s_{n-1}, \ldots, s_1 \mid n)$. And this will be obtained from the product of conditional distributions. Hence, on taking the product and summing over k and j. We find then that the density

$$\sum_{k=0}^{n-1} \pi_k^T \cdot f_n^*(s_n, s_{n-1}, \ldots, s_1 \mid k) +$$

$$\pi_{n+}^T \cdot \mathbf{f}_n^*(s_n, s_{n-1}, \ldots, s_1 \mid n)$$

i.e., the joint probability density function is the product of the marginal p.d.f.'s.
From these considerations it easily follows that the marginal distribution of D has Laplace transform when $\frac{\lambda}{\mu} < 1$,i.e. ,
$\pi_0^T \cdot \mathbf{a}^*(s_1) + \pi_{1+}^T(H^*(S_1), H^*(S_1), \ldots, H^*(S_1))^T$

4 Autocovariance of interdeparture times

We shall first derive the stationary probability π at departure point in $K_M/G/1$. Then we expect to obtain $Cov[D_1, D_n]$, for $1 < n < \infty$. We define it to be weakly stationary if the expected value of D_t is a constant for all t. The covariance matrix of $(D_1, D_2, \ldots, D_n)$ for all nonempty finite sets of indices denoted by $(t_1, t_2, \ldots t_n)$ and all m such that $t_1, t_2, \ldots, t_n, t_1+h, t_2+h, \ldots, t_n+h$ are contained in the index set. The covariance of D_1 and D_{1+m} depends only on the distance m and we may write $\gamma(m) = Cov[D_1, D_{1+m}]$,where

$$Cov[D_1, D_n] =$$

$$\frac{\partial^2 f_n(s_n, s_{n-1}, \ldots, s_1)}{\partial s_n \partial s_1} |_{s_n = s_{n-1} = \ldots = s_1 = 0} - (E[D])^2$$

The autocorrelation of lag 1, is written as , m=1. It is not worth writing out the precise form of $f(D_{n-1}\ , D_n)$ in the special Erlangian case. Instead, a measure of the dependence in the departure process will be found by taking the expectation of $(D_{n-1}\ , D_n)$.

5 The correlation structure

By the recursive procedure given in section 3, we can calculate the Laplace transform $f_n(s_n, s_{n-1}, \ldots, s_1)$ for the joint distribution of n consecutive interdeparture times $D_1, D_2, \ldots, D_n$. Such calculation is made possible by symbolic formula manipulation software Mathematica [14]. We can then obtain the covariance of D_1 and D_n.

In this section, we present the result for a $GI/G/1$ queue. According to Takagi and Nishi [15], $Cov[D_1, D_n]$ is obtained for an $M/G/1$ queue. We display the correlation coefficients of the interdeparture times defined by $\gamma(m)$ for the Erlang-K distribution of the service time which has the unit mean and squared which is independent of the arrival rate λ . Takegi and Nishi [12] gives the expression for $\gamma(3)$ for the $M/G/1$ queue, which agrees with our special case. In Fig.1, we plot $\gamma(m)$ for $m = 1, 2, \ldots, 5$ respectively, for $E_2/E_2/1$,$E_3/E_2/1$,$E_4/E_2/1$,$E_5/E_2/1$. We observe that $\gamma(m)$ is always nonnegative. Thus, we always have $lim \gamma(m) = 0$. Given n and m finite, $\gamma(m)$ is a unimodal function of λ . Given λ, $r(m)$ increases with K and decreases with m.

	D_1	D_2	D_3
K=2	-4.98328	-4.98328	-4.98328
K=3	-4.99494	-4.99494	-4.99494
K=4	-4.99792	-4.99792	-4.99792
K=5	-4.99905	-4.99905	-4.99905

	D_1, D_2	D_1, D_3	$\gamma(1)$	$\gamma(2)$
K=2	24.7007	24.9001	-0.2993	-0.0999
K=3	24.6259	24.9476	-0.3741	-0.0524
K=4	24.5858	24.9666	-0.4142	-0.0334
K=5	24.5626	24.9762	-0.4374	-0.0238

6 The Variability of the Output Process

One additional measure of interest is the variability of the output process. In general the more variable the output process is, the greater the amount of work-in-process will be present in downstream subsystems. Two possible measures for the variability of the output process are the variance of the interdeparture distribution and the *asymptotic variance*. The asymptotic variance was introduced in a sequence of papers by Whitt [16,17], Albin [1] and Hendricks [9]. The asymptotic variance is the limiting variance, per departure, of the time of the nth departure (measured arbitrarily from a departure epoch). Mathematically, the asymptotic variance(A Var) is give by:

$$\text{AVar} = \lim_{n \to 0} \frac{1}{n} \text{Var}(\sum_{j=0}^{n-1} D_j)$$

Now by Assumption of (the process is stationary) and by assuming the lag j correlations are zero for $j \geq q$, where q is finite, the asymptotic variance reduces to

$$\text{AVar} \approx \text{Var}(D)(1 + 2\sum_{j=1}^{q} \text{Corr}_j).$$

Therefore, the asymptotic variance can be obtained from the variance of the interdeparture distribution and the correlation structure.

7 Conclusion

In this report, we present an efficient numerical method for calculation the waiting time and idle time distributions of the algorithmic $GI/G/1$ queue. Our method is based on the Matrix-algebraic techniques, which is briefly reviewed. Compared to the related methods suggested in the literature, our method seems to perform very well, and it is often faster by several orders of magnitude. A number of numerical examples conclude the paper.

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