行政院國家科學委員會專題研究計畫　成果報告

具遞增損害率之時間特性的流程分析模式

計畫類別：個別型計畫
計畫編號：NSC93-2213-E-004-006-
執行期間：93 年 08 月 01 日至 94 年 07 月 31 日
執行單位：國立政治大學應用數學學系

計畫主持人：陸行

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 94 年 10 月 3 日

行政院國家科學委員會補助專題研究計畫成果報告

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計畫類別：• 個別型計畫　□整合型計畫

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執行單位：政治大學應用數學系

中　華　民　國　94　年　8　月　2　日

行政院國家科學委員會專題研究計畫成果報告
具遞增損害率之時間特性的流程分析模式
An Analytical Model for Service Systems with Increasing Failure Rate Processing Times

計畫編號: **NSC 93-2213-E-004-006**
執行期限: 93年08月01日至94年07月31日
主持人:陸行 政治大學應用數學系

1 Abstract

Our goal is to investigate the output process under what conditions the interdeparture time will preserve the IFR property. Because of the complexity of the stationary probability density, we take advantage of computer to visualize the performance of the output process. We found the interdeparture time doesn't always preserve the IFR property even if the interarrival time and service time are Erlang distributions with IFR. We give several theoretic analysis and present some numerical results of $E_m/E_k/1$ queues. For our experiment, if $m \geq k$, the interdeparture time of $E_m/E_k/1$ remains the IFR property.

(Keywords: Departure Process, $PH/G/1$,Hazard rate, IFR.)

中文摘要

而在本篇論文中，我們研究$PH/G/1$模型的輸出過程。首先我們建構輸出間隔機率分配的LST轉換式，並給定一些分析輸出過程的指標，如輸出間隔的平均值、變異數和變異係數。特別是分析輸出間隔的IFR性質，我們目的在於討論在何種條件下其輸出間隔保有IFR特性。由於系統穩態機率分配的複雜性，我們藉由電腦協助演算$E_m/E_k/1$的輸出間隔並展示其數值結果。我們發現即使到達間隔及服務時間均具有IFR性質，其輸出間隔也未必保有IFR特性。然而在我們的實驗中，我們發現對於$E_m/E_k/1$模型中，當m大於或等於k時，其輸出間隔保有IFR性質。

(關鍵字:輸出過程, $PH/G/1$,Hazard rate, IFR.)

2 Introduction

For decades, queueing networks has been basic analytic model for the study of computers, communication networks and manufacturing systems. Recently, the study of the networks shifted their focus to the departure processes in the queueing systems, because the departure processes from one server may be the input or arrival processes of another server. Therefore, it is important to characterize the departure processes in the queueing systems.

From the pass studies, we know a $GI/G/1$ queue has output instants which form a renewal process if and only if the arrival process is Poisson and the services times are exponential. When the output process is not a renewal process, researchers studied the correlation structure of the output process. This is because departure processes of queues other than $M/G/1$ are difficult to characterize. In this paper, our goal is to investigate what properties will be preserved for the departure process if the queue is no more $M/M/1$ queue. We will consider a $PH/G/1$ queue and construct the Laplace-Stieltjes transform (LST) of the interdeparture time distribution, where

PH denotes the phase type distribution proposed by Neuts [11, 13, 14, 15]. The phase type distribution can be a good approximation for general distribution. We analyze the stochastic properties, such as Increasing Failure Rate (IFR) for the interdeparture time. Because of the computational complexity of $PH/G/1$ and the stationary probability density of the number of customers in the system, we restrict our numerical examples to the $E_m/E_k/1$ queue, where both the interarrival time and service time are Erlang distributions. We illustrate some numerical results and consider under what conditions the interdeparture time is IFR in the $E_m/E_k/1$ queue.

Since the 1950's, many researchers [1, 3, 7, 8] have studied the output process of queueing models. They focused on the distribution of the interdeparture time and the correlation structure of the output process. In this paper, we use an alternative approach to analyze the performance of the output process. We consider the stochastic order relations of the interdeparture time, such as IFR. They are important for comparison of "new" and "residual" life times [2, 4]. The stochastic order relations of the output processes from one server may be important indices for the input processes of another server in the network [5,6]. In this study, we investigate under what conditions the interdeparture time will preserve the IFR property. We take advantage of computer to conduct some experiments and visualize the failure rate of the interdeparture time of $E_m/E_k/1$ queues

3 Performance analysis

Now, we consider some important performance measures of the interdeparture time for $E_m/E_k/1$ queue.

Lemma 3.1 *For the stationary $E_m/E_k/1$ queue, we have*

$$E[D] = \sum_{j=1}^{m} \frac{\boldsymbol{\pi}_0(j)(m+1-j)}{m\lambda} + \frac{1}{\mu}.$$

Lemma 3.2 *For the stationary $E_m/E_k/1$ queue, we have the variance of the interdeparture time $Var[D]$ is*

$$\sum_{j=1}^{m} \frac{\boldsymbol{\pi}_0(j)(m+1-j)^2}{(m\lambda)^2} + \sum_{j=1}^{m} \frac{\boldsymbol{\pi}_0(j)(m+1-j)}{(m\lambda)^2}$$

$$-(\sum_{j=1}^{m} \frac{\boldsymbol{\pi}_0(j)(m+1-j)}{m\lambda})^2 + \frac{1}{k\mu^2}.$$

Lemma 3.3 *For the stationary $E_m/E_k/1$ queue, we have $E[D] = \frac{1}{\lambda}$ which implies*

$$\sum_{j=1}^{m} \boldsymbol{\pi}_0(j)(m+1-j) = m(1-\frac{\lambda}{\mu}).$$

Theorem 3.4 *For a stationary $E_m/E_k/1$ model, we have the square coefficient of variation of the interdeparture time $c_D^2 = \frac{Var[D]}{E^2[D]} \leq 1$ and*

$$c_D^2 = \frac{1}{m^2}\sum_{j=1}^{m} \boldsymbol{\pi}_0(j)(m+1-j)^2$$

$$+\frac{1}{m}(1-\rho) - (1-\rho)^2 + \frac{1}{k}\rho^2$$

By a different approach, Buzacott and Shanthikumar [2] showed that for any $GI/G/1$ queue with DMRL(decreasing mean residual life)interarrival time, the upper bound on c_D^2 is $c_D^2 \leq c_A^2(1-\rho) + \rho^2 c_S^2 + \rho(1-\rho)$, and when the interarrival time have IMRL(increasing mean residual life) property, the lower bound on c_D^2 is $c_D^2 \geq c_A^2(1-\rho) + \rho^2 c_S^2 + \rho(1-\rho)$, where c_A^2 and c_S^2 are the the square coefficient of variation of the interarrival time and service time, respectively. Since Erlang distributions are DMRL and hyper-exponential distributions are IMRL, it is easy to check with the upper bound condition of the $E_m/E_k/1$ queue that has $c_D^2 \leq 1$. With the lower bound condition, it is easy to show that the $H_m/H_k/1$ queue has $c_D^2 \geq 1$ in which the interarrival time and service time are both hyper-exponential distributions with c_A^2 and c_S^2 being greater than one.

4 Stochastic properties

In this subsection, we consider the failure rate for stationary interdeparture time of $E_m/E_k/1$ queue. We find that the stationary interdeparture time does not preserve the property of IFR even if the interarrival time and the service time are both IFR. We will consider under what conditions the interdeparture time preserve the property of IFR for the $E_m/E_k/1$ queue.

we employ the method of partial fractions that separates it in terms of λ and μ respectively [9,10, 12]. We have

$$\tilde{D}(s) = \sum_{j=1}^{m} a_j (\frac{m\lambda}{s+m\lambda})^j + \sum_{i=1}^{k} b_i (\frac{k\mu}{s+k\mu})^i, \qquad (1)$$

where a_j and b_i are coefficients associated with each term of λ and μ. We will discuss the method of partial fractions and how to obtain a_j and b_i in numerical examples.

Taking the inverse transform of above equation, we have the density function of the interdeparture time is in the form of

$$d(x) = \sum_{j=1}^{m} a_j \frac{(m\lambda)^j}{(j-1)!} x^{j-1} exp(-m\lambda x) \qquad (2)$$

$$+ \sum_{i=1}^{k} b_i \frac{(k\mu)^i}{(i-1)!} x^{i-1} exp(-k\mu x). \qquad (3)$$

The coefficients a_j and b_i are attained in accordance to the method of partial fractions and the stationary probability $\boldsymbol{\pi}_0$ that a departure leaves the system empty with respect to the arrival phases. First we consider the initial value of the failure rate $r(x)$ of the interdeparture time distribution and we have the following theorem.

Theorem 4.1 *For the stationary $E_m/E_k/1$ queue, the initial value of the failure rate $r(x)$ of the interdeparture time distribution is*

$$\lim_{x\to 0^+} r(x) = \begin{cases} (1-\boldsymbol{\pi}_0\mathbf{e})\mu & if \;\; k=1 \\ 0 & if \;\; k \geq 2. \end{cases}$$

where $\boldsymbol{\pi}_0$ is the stationary probability that a departure leaves the system empty with respect to different arrival phases.

Now we consider the final value of the failure rate $r(x)$ of the interdeparture time distribution. Since the final value of failure rate of Erlang-k distribution converges to θ as $x \to \infty$, where $\theta = k\lambda$, does the failure rate of the interdeparture time distribution $r(x)$ converge as $x \to \infty$ and what is the limit?

Theorem 4.2 *For the stationary $E_m/E_k/1$ queue, the final value of the failure rate $r(x)$ of the interdeparture time distribution is given by*

$$\lim_{x\to\infty} r(x) = \begin{cases} m\lambda & if \;\; m\lambda \leq k\mu, \quad (k \geq m\rho) \\ k\mu & if \;\; m\lambda > k\mu, \quad (k < m\rho). \end{cases}$$

That is

$$\lim_{x\to\infty} r(x) = \min\{m\lambda, k\mu\}.$$

5 Hazard rate analysis of $E_m/D/1$ queues

In this subsection, we consider a deterministic service case. Since the Erlang-k distribution converges to a constant value as $k \to \infty$. According to Section 2, we can obtain several performance indices for the departure process. Now we examine whether or not the interdeparture time of $E_m/D/1$ queue has non-decreasing failure rate. By Theorem **??**, the failure rate $r(x)$ of the stationary $E_m/D/1$ queue is given by (**??**) where $h = 1/\mu$ and $r_I(\cdot)$ is the failure rate of the idle time distribution $I(\cdot)$. The LST of $I(\cdot)$ is given by $\tilde{I}(s) = \sum_{j=1}^{m} \boldsymbol{\pi}_0(j)(\frac{m\lambda}{s+m\lambda})^{m+1-j} + (1-\boldsymbol{\pi}_0\mathbf{e})$. Let $i(x)$ be the probability density function of $I(x)$. We have

$$\lim_{x\to 0^+} I(x) = 1 - \boldsymbol{\pi}_0\mathbf{e} \qquad (4)$$

and

$$\lim_{x\to 0^+} i(x) = \boldsymbol{\pi}_0(m)m\lambda. \qquad (5)$$

Since $r_I(x) = \frac{i(x)}{1-I(x)}$, we have

$$\lim_{x\to 0^+} r_I(x) = \frac{\boldsymbol{\pi}_0(m)m\lambda}{\boldsymbol{\pi}_0\mathbf{e}}. \qquad (6)$$

For $m = 1$, namely, $M/D/1$ queue, we have $I(x) = 1-(1-\rho)exp(-\lambda x)$ for $x > 0$. Then, $\lim_{x\to 0^+} r_I(x) =$

λ. When we let $\lambda = \rho$ with service rate $\mu = 1$, we have $\lim_{x\to h^+} r(x) < \rho/(1-\rho)$. This implies that the output process for $M/D/1$ queue doesn't have IFR property.

If $m \geq 2$, we have $\lim_{x\to h^+} r(x) = \lim_{x\to 0^+} r_I(x) = \frac{\boldsymbol{\pi}_0(m)m\lambda}{\boldsymbol{\pi}_0\mathbf{e}}$. We will discuss whether $\lim_{x\to h^+} r(x)$ is larger than $(1-\boldsymbol{\pi}_0\mathbf{e})/\boldsymbol{\pi}_0\mathbf{e}$ or not by numerical method in next section.

6 Conclusions

In this project, we derived the Laplace-Stieltjes transform (LST) of the interdeparture time of $PH/G/1$ queue and gave some indices for the performance analysis of the departure process of $PH/G/1$ queue, such as the moments, the variance, and the square coefficient of variation. We showed the $E_m/E_k/1$ queue has $c_D^2 \leq 1$ and the $H_m/H_k/1$ queue has $c_D^2 \geq 1$.

Especially, we analyzed the failure rate of the stationary interdeparture time. To the best of our knowledge, it has not been studied before in this aspect. We focused on the IFR property of the interdeparture time of $E_m/E_k/1$ queue. Because of the complexity of the stationary probability density $\boldsymbol{\pi}_0$, we took advantage of computer to visualize the performance of the output process. We found the interdeparture time doesn't always preserve the IFR property even if the interarrival time and service time are Erlang distributions with IFR. But if $k \leq m$, the interdeparture time of $E_m/E_k/1$ remains the IFR property in our experiments.

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