行政院國家科學委員會專題研究計畫 成果報告

非線性擾動下三維有界域半線性波方式爆炸解之穩定性研究
研究成果報告(精簡版)

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 99-2115-M-004-001-
執 行 期 間 ： 99 年 08 月 01 日至 100 年 07 月 31 日
執 行 單 位 ： 國立政治大學應用數學學系

計 畫 主 持 人 ： 李明融
共 同 主 持 人 ： 謝宗翰、白仁德

報 告 附 件 ： 國外研究心得報告

處 理 方 式 ： 本計畫涉及專利或其他智慧財產權，2 年後可公開查詢

中 華 民 國 100 年 09 月 23 日

Stability of positive solutions for some semilinear wave equations under nonlinear perturbation near blow-up solutions in 3-space dimension

$$\Box\ \mathrm{u} - \mathrm{u}^p + \lambda \mathrm{u}^q = 0$$

Meng-Rong Li

Department of Mathematical Sciences National Chengchi University

Abstract In this research we treat the stability of positive solutions of some particular semilinear wave equations under nonlinear perturbation in bounded domain near blow-up solutions in 3-space dimension.

1. Introduction

Consider the initial value problem for the semilinear wave equation of the type

$$\Box\, u + g(u) = 0 \quad in \quad [0,T) \times \mathbb{R}^3, \tag{0.1}$$

$$u(0,\cdot) = u_0,\ \dot{u}(0,\cdot) = u_1, \tag{0.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a real valued function, the initial data are given sufficiently smooth functions and $\Box u := u_{tt} - \triangle u$, $\triangle$ is the Laplace operator. The linear case $g(u) = mu$, where m is a constant, corresponds to the classical Klein Gordon equation in relativistic particle physics; the constant m is interpreted as the mass and is assumed to be nonnegative generally. To model also nonlinear phenomena like quantization, in the 1950s equations of (0.1) type with nonlinearities like $g(u) = mu + u^3$; $m \geq 0$; were proposed as models in relativistic quantum mechanics with local interaction. Solutions could be considered as real or complex valued functions. In the latter case it was assumed that the nonlinearity commutes with the phase; that is, $g\left(e^{i\varphi}u\right) = \varepsilon e^{i\varphi} g(u)$ for $\varphi \in \mathbb{R}$ and that $g(0) = 0$. In this case, g may be expressed $g(u) = uf\left(|u|^2\right)$, which gives the study of equation (0:1) $[J]$. In the noncoercive case it is easy to construct solutions of (0.1) with smooth initial data that blow up in finite time; for instance, for any $\alpha > 0$ the function $u(t;x) = (1-t)^{-1/m}$ solves the equation $\Box u + \alpha(1+\alpha)u\,|u|^{2m} = 0; m \in \mathbb{N}$ and blows up at $t = 1$. Modifying the initial data off $\{x : |x| \leq 2\}$, say, we even posses a singular solution with C^{∞}-data having compact support.

In this study we want to deal with the stability of positive solutions for the semilinear wave equation

$$(1.1) \qquad \Box\ u = u^p + \lambda u^q \text{ in } [0,T) \times \Omega, \Omega \subset \mathbb{R}^3$$

with boundary value null and initial values $u(0,\cdot) = u_0(\cdot) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\dot{u}(0,\cdot) = u_1(\cdot) \in H_0^1(\Omega)$, where $p, q \in (1,\infty)$ and Ω is a bounded domain in $\mathbb{R}^3$.

We will use the following notations:

$$\cdot := \frac{\partial}{\partial t}, Du := (\dot{u}, \nabla u), \triangle u := \frac{\partial^2 u}{\partial^2 x_1} + \frac{\partial^2 u}{\partial^2 x_2} + \frac{\partial^2 u}{\partial^2 x_3},$$

$$a(t) := \int_\Omega u^2(t,x)\,dx, E_\lambda(t) := \int_\Omega \left(|Du|^2 - \frac{2}{p+1}u^{p+1} - \frac{2\lambda}{q+1}u^{q+1}\right)(t,x)\,dx.$$

For a Banach space X and $0 < T \leq \infty$ we set

$$C^k(0,T,X) = \text{Space of } C^k - \text{ functions} : [0,T) \to X,$$

$$H1 := C^1\left(0,T,H_0^1(\Omega)\right) \cap C^2\left(0,T,L^2(\Omega)\right).$$

The existence result to the equation (1.1) is proved $[Li\ 3]$ and the positive solution blows-up in finite time if $\lambda \geq 0$ $[Li\ 2]$, this means that the positive solutions for the semilinear wave equation

$$(1.2) \qquad \Box\ u = u^p \text{ in } [0,T) \times \Omega,$$

$$u(0,\cdot) = u_0(\cdot) \in H^2(\Omega) \cap H_0^1(\Omega),$$

$$\dot{u}(0,\cdot) = u_1(\cdot) \in H_0^1(\Omega),$$

is stable under nonlinear perturbation λu^q providing $p > 1, q > 1, \lambda > 0$; but it is not clearly whether it is also true for any $p > 1, q > 1, \lambda < 0$? If so, we would want to estimate the blow-up time and the blow-up rate under such a situation.

It is also important to study the asymptotic behavior of the solution u_λ; the velocity and the rate of the approximation for λ approaches to zero.

Such questions are also not easy to answer even under the case for the ordinary differential equation

$$(1.3) \qquad u'' = u^p\left(c + \lambda u'(t)^q\right),$$

$$u(0) = u_0, u'(0) = u_1,$$

where $p > 1, q > 1, c > 0, \lambda > 0$. We have studied the blow-up behavior of the solution for problem (1.3) and got some estimates on blow-up time and blow-up rate [Li4] but it is difficult to find the real blow-up time (life-span). Further literature could be fund in [S], [R], [W1] and [W2].

In this study we hope that our ideals used in $[Li\,2]$, $[Li\,4]$, $[Li\,5]$,$[Li\ 7]$, $[Li8]$, $[LiLinShieh]$, $[ShiehLi]$ and $[SLLLW]$ can do help us dealing such problem (1.1) on our topics.

2. **Definition and Fundamental Lemma**

There are many definitions of the weak solutions of the initial-boundary problems of the wave equation, we use here as following.

Definition 2.1: *For $p>1$, $u\in H1$ is called a positive weakly solution of equation* (1.1)*, if*

$$\int_0^t\int_\Omega\left(\dot{u}\left(r,x\right)\dot{\varphi}\left(r,x\right)+\left(u^p+\lambda u^q\right)\left(r,x\right)\varphi\left(r,x\right)\right)dxdr$$

$$=\int_0^t\int_\Omega\nabla u\left(r,x\right)\cdot\nabla\varphi\left(r,x\right)dxdr\ \forall\varphi\in H1$$

and

$$\int_0^t\int_\Omega u\left(r,x\right)\psi\left(r,x\right)dxdr\geq 0$$

for each positive $\psi\in C_0^\infty\left([0,T)\times\Omega\right)$.

Remark 2.2:

1) The definition 1.1 is resulted from the multiplying with φ to the equation (1.1) and integrating in Ω from 0 to t.

2) From the local Lipschitz functions $u^p+\lambda u^q, p>1, q>1$ the initial-boundary value problem (1.1) possesses a unique solution in $H1$ [Li1]. Hereafter we use the notations:

$$\frac{1}{C_\Omega}:=\eta_1=\sup\left\{\left\|u\right\|_2/\left\|\frac{\partial u}{\partial x}\right\|_2:u\in H_0^1\left(\Omega\right)\right\},$$

$$\lambda_q\ =\sup\left\{\left\|u\right\|_q/\left\|\frac{\partial u}{\partial x}\right\|_2:u\in H_0^1\left(\Omega\right)\cap L_q\left(\Omega\right)\right\},q>1.$$

In this study we need the following lemmas

Lemma 2.3: *Suppose that $u\in H1$ is a weakly positive solution of* (1.1) with $E_\lambda\left(0\right)=0$ for $p>1, q>1$, *then for $a\left(0\right)>0$ we have:*

(i) *$a\in C^2\left(\mathbb{R}^+\right)$ and $E_\lambda\left(t\right)=E_\lambda\left(0\right)\quad\forall t\in[0,T)$.*

(ii) *$a'\left(t\right)>0\quad\forall t\in[0,T)$, provided $a'\left(0\right)>0$.*

(iii) *$a'\left(t\right)>0\quad\forall t\in(0,T)$, if $a'\left(0\right)=0$.*

(iv) *For $a'\left(0\right)<0$, there exists a constant $t_0>0$ with*

$$a'\left(t\right)>0\quad\forall t>t_0$$

and $a'\left(t_0\right)=0$.

Lemma 2.4: *Suppose that u is a positive weakly solution in $H1$ of equation (1.1) with $u(0,\cdot) = 0 = \dot{u}(0,\cdot)$ in $L^2(\Omega)$. For $p > 1, q > 1, \lambda > 0$, we have $u \equiv 0$ in $H1$.*

According to Lemma 2.4, we discuss the following theme

(3) $E_\lambda(0) = 0, a(0) > 0$ and $a'(0) \geq 0$ or $a'(0) < 0$.

(4) $E_\lambda(0) < 0, a(0) > 0$ and $a'(0) \geq 0$ or $a'(0) < 0$.

3. **Estimates for the Life-Span**

3.1. **Estimates for the Life-Span of the Solutions of (1.1) under Null-Energy.** We study the case that $E_\lambda(0) = 0,\ p > 1, q > 1, \lambda > 0$ and divide it into

two parts

(i) $a(0) > 0, a'(0) \geq 0$ and (ii) $a(0) > 0, a'(0) < 0$.

Remark 3.. 1) The local existence and uniqueness of solutions of equation (1.1) in $H1$ are known [Li 2].

2) For $\lambda = 0$, x $p > 1$ and $E_\lambda(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (1.1) is bounded by

$$T \leq \alpha_1 := k_2^{-1} \sin^{-1}\left(\frac{k_2}{k_1 a^{\frac{p-1}{4}}(0)}\right)$$

with

$$k_1 := \frac{p-1}{4} \cdot a^{-\frac{p-1}{4}}(0)\sqrt{a'(0)\, a^{-2}(0) + 4C_\Omega^2},\ k_2 := \frac{p-1}{2} C_\Omega,$$
$$\frac{1}{C_\Omega} := \eta_1 = \sup\left\{\|u\|_2 / \|Du\|_2 : u \in H_0^1(\Omega)\right\}.$$

3.2. **Estimates for the Life-Span of the Solutions of equation (1.1) under Negativ-Energy.** We use the following result and those argumentations of proof are not true for positive energy, so under positive energy we need another method to show the similar results.

Lemma 3: *Suppose that $u \in H1$ is a positive weakly solution of equation (1.1) with $a(0) > 0$ and $E_\lambda(0) < 0$ for $\lambda = 0$. Then*

(i) for $a'(0) \geq 0$, we have $a'(t) > 0\ \ \forall t > 0$.
(ii) for $a'(0) < 0$, there exists a constant $t_5 > 0$ with

$$a'(t) > 0\ \ \forall t > t_5,\ a'(t_5) = 0$$

and

$$t_5 \leq t_6 := \frac{-a'(0)}{(p-1)(\delta^2 - E_\lambda(0))},$$

where δ is the positive root of the equation

$$\frac{2}{p+1} \lambda_{p+1}^{p+1} \cdot r^{p+1} - r^2 + E_\lambda(0) = 0.$$

4. **Stability of positive solutions of equation (1.1) near blow-up solutions under Negativ-Energy**

In this study we use our ideals used in $[Li\ 2]$, $[Li\ 4]$, $[Li\ 5]$,$[Li\ 7]$, $[Li8]$, $[LiLinShieh]$, $[ShiehLi]$ and $[SLLLW]$ to deal such problem (1.1) on our topics under negative energy and obtain the following results:

Theorem 4.1: *Suppose that* $u_\lambda \in H1$ *is a weakly positive solution of* (1.1) with $E_\lambda(0) \leq 0$ for $p > 1, q > 1$, *then for* $a(0) > 0$ *we have*:

The equation (1.1) *is stable for* $\lambda \to 0^+$; *this means that weakly positive solution* u_λ *of* (SL) *blows up in finite time for* $\lambda \to 0^+$.

Theorem 4.2: *Suppose that* $u_\lambda \in H1$ *is a weakly positive solution of* (1.1) with $E_\lambda(0) \leq 0$ for $p > 1, q > 1$, *then for* $a(0) > 0$ *we have*:

The equation (1.1) *is stable for* $p > q$, $\lambda \to 0^-$; *this means that weakly positive solution* u_λ *of* (1.1) *blows up in finite time for* $p > q$, $\lambda \to 0^-$.

Theorem 4.3: *Suppose that* $u_\lambda \in H1$ *is a weakly positive solution of* (1.1) with $E_\lambda(0) \leq 0$ for $p > 1, q > 1$, *then for* $a(0) > 0$ *we have*:

The equation (1.1) *is unstable for* $p < q$, $\lambda \to 0^-$; *this means that some weakly positive solution* u_λ *of* (1.1) *blow up in finite time for* $p < q$, $\lambda \to 0^-$; *but also there were some global weakly positive solution* u_λ *of* (1.1) *for* $p < q$, $\lambda \to 0^-$.

Remark:

The decade rate of the difference of life-spans T_λ of u_λ and T of u, can not be estimated very well for $\lambda \to 0$; thus it will be a good topic on asymptotic behavior near the blow-up solutions.

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一月十七日(一)　與尤釋賢教授 討論 波方程式解的一些性質 (I)

一月十八日(二)　與尤釋賢教授 討論 波方程式解的一些性質 (II)

一月十九日(三)　與尤釋賢教授 討論 波方程式解的奇異性 (I)

一月二十日(四)　與尤釋賢教授 討論 波方程式解的奇異性 (II)

一月二十一日(五)　與尤釋賢教授 討論 波方程式解的奇異集 (I)

一月二十二日(六)　與尤釋賢教授 討論 波方程式解的奇異集 (II)

一月二十三日(日)　與尤釋賢教授 共同討論行進波與震波之交互作用(I)

一月二十四日(一)　與尤釋賢教授 共同討論行進波與震波之交互作用(II)

並贈語曰

星州晚暮

侵晨蘇醒夜朦朧　美人依舊臨虛空
埜花不知寄何處　星州大學綠萬叢
細索問題本原地　菩提知慧無底洞
醉月早忘千年愁　與君共享夕陽紅

神奇無雨

今日最神奇　此間無煙雨　禿筆忘畊雲　宿墨離塵汙
星州新夜月　行者問所須　勸君壹壺酒　千憂自茲去

庚寅冬禿筆風語于星州怡閣

國科會補助計畫衍生研發成果推廣資料表

日期:2011/09/20

國科會補助計畫	計畫名稱: 非線性擾動下三維有界域半線性波方式爆炸解之穩定性研究
	計畫主持人: 李明融
	計畫編號: 99-2115-M-004-001- 學門領域: 微分方程

無研發成果推廣資料

99 年度專題研究計畫研究成果彙整表

計畫主持人：李明融			計畫編號：99-2115-M-004-001-			
計畫名稱：非線性擾動下三維有界域半線性波方式爆炸解之穩定性研究						
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）

成果項目			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比	單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
國內	論文著作	期刊論文	1	1	100%	篇	
		研究報告/技術報告	1	1	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	3	0	100%	人次	
		博士生	2	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	2	2	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無法詳盡

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   ■達成目標

   □未達成目標（請說明，以 100 字為限）

   　　□實驗失敗

   　　□因故實驗中斷

   　　□其他原因

   說明：

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：□已發表 □未發表之文稿 ■撰寫中 □無

   專利：□已獲得 □申請中 ■無

   技轉：□已技轉 □洽談中 ■無

   其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）